THE

ELEMENTS OF EUCLID

THE FIRST SIX BOOKS

AND

THE ELEVENTH AND TWELFTH

FROM THE TEXT OF

ROBERT SIMSON, M.D.

EDITED, IN THE SYMBOLICAL FORM

BY

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NEW EDITION

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TO THE

FORMER EDITION.

The present Edition of the Elements of Euclid is printed, with a few variations, from the text of Dr. Robert Simson. These variations may be included under the following heads:—

1. The Enunciations, which in the modern editions of Simson's Euclid are expressed in the present tense, are here given in the future.

2. In the Problems, the demonstration and the construction, if there be any belonging to it, are separated from that part of the proposition which forms the actual solution of the problem.

3. In some of the propositions, which are divided into two or more cases, a slight alteration has been made, in order to include them all under one general enunciation.

4. In a few instances, where there appeared t be any obscurity in the Demonstration, whic could be removed by a transposition of the sen tence, or by the introduction of a step, the Edito has ventured to make the alteration.

5. Numerous marginal references are insertec in addition to those which appeared in forme editions. Where a simple reference has not bee sufficient to make the step clear, a short note ha been introduced at the bottom of the page.

6. The punctuation has been corrected, partl by a reference to the 4to Edition of 1756, an partly by the Editor's own judgment.

R. N. ADAMS.

Christ's Hospital,
Sept. 1824.

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TO THE

NEW EDITION.

In the Universities, algebraical and geometrical Symbols have now been so generally adopted, not only in MSS., but also in Works issuing from the Press, that it seems scarcely necessary to adduce any other reason for the use of them in a new Pocket Edition of the Elements of Euclid; that Work, to which, perhaps, of all others, the symbolical Notation is most eminently applicable.

With the algebraical Symbols introduced, the mathematical reader will already be perfectly familiar; and of the geometrical, there are but two (those for the straight line and parallelopiped) which have not long been in common use.

The text of the former edition of the work has been adhered to, with such slight variations only as were rendered necessary by the nature of the plan, the principal feature of which was to exhibit the Propositions under the form in which they are usually written by students in the University.

R. BLAKELOCK.

Catharine Hall, Cambridge,
Jan. 8. 1831.

SYMBOLS.

Algebraical.

$\because$ because	$\therefore$ therefore

In the use of the signs of equality and inequality a slight discrepancy will be observed in regard to the introduction of the auxiliary verbs *is*, *are*, &c.; the symbol $=$ has been used, as in fact the word *equal* itself is, both adjectively and as a verb; before the signs $>$ and $<$ the auxiliary verb has generally been expressed; and this has always been done in each case, when the omission of it might lead to any ambiguity.

$=$ equal	$\ngtr$ not greater than
$\neq$ not equal	$\nless$ not less than
$>$ greater than	$+$ plus, the sign of addition
$<$ less than	$-$ minus, the sign of subtraction

AB. CD AB multiplied into CD; it is also used to represent the rectangle contained by the two straight lines AB and CD as the sides.

A : B :: C : D signifies that the ratio of A to B is the same with the ratio of C to D: and is read, as A is to B, so is C to D; or A is to B, as C is to D.

Geometrical.

| straight line	$\angle$ angle	$\odot$ circle
$\parallel$ parallel to	$\triangle$ triangle	$\odot^{ce}$ circumference
$\parallel^{s}$ parallels	▱ parallelogram	
$\perp$ perpendicular to	parallelopiped	

When, in the former Edition the word *circumference* has been used to express only part of the whole circular boundary the term *arc* has been introduced instead of the symbol $\odot^{ce}$.

ABBREVIATIONS.

alt.	alternate	altit.	altitude

bis^t - - - bisect

The active participle *bisecting* is represented by bis^g, the past participle *bisected* by bis^d; and similarly in the other abbreviated verbs.

circumsc.	circumscribe	opp.	opposite
com.	common	prod.	produce
constr^n	construction	prop^n	proposition
cyl.	cylinder	p^t	point
desc.	describe	pntg^n	pentagon
dist.	distance	pyr^d	pyramid
div.	divide	quadrilat^l	quadrilateral
dupl.	duplicate	r^o	ratio
equiang^r	equiangular	rect.	rectangle
equilat^l	equilateral	rect^l	rectilineal
ext.	exterior	rem^r	remainder
extr^y	extremity	r^t	right
homol.	homologous	seg^t	segment
hxg^n	hexagon	sq.	square
int.	interior	tripl.	triplicate
magn.	magnitude	w^h	which
n^o	number		

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I.

A POINT is that which hath no parts, or which hath no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which hath only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

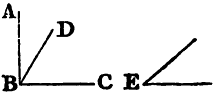
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

" A plane angle is the inclination of two lines to " one another in a plane, which meet together, " but are not in the same direction."

IX.

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N.B. 'When several angles are at one point B, 'any one of them is expressed by three letters, of 'which the letter that is at the vertex of the angle, 'that is, at the point in which the straight lines 'that contain the angle meet one another, is put 'between the other two letters, and one of these 'two is somewhere upon one of those straight lines, 'and the other upon the other line: Thus the angle 'which is contained by the straight lines, AB, CB, 'is named the angle ABC, or CBA; that which is 'contained by AB, DB, is named the angle ABD, 'or DBA; and that which is contained by DB, 'CB, is called the angle DBC, or CBD; but if 'there be only one angle at a point, it may be 'expressed by a letter placed at that point: as the 'angle at E.'

X.

When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle;

and the straight line which stands on the other is called a perpendicular to it.

XI.

An obtuse angle is that which is greater than a right angle.

XII.

An acute angle is that which is less than a right angle.

XIII.

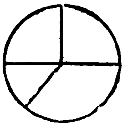
"A term or boundary is the extremity of any thing."

XIV.

A figure is that which is inclosed by one or more boundaries.

XV

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



XVI.

And this point is called the centre of the circle.

XVII.

A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

"A segment of a circle is the figure contained by a "straight line, and the circumference it cuts off."

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines.

XXII.

Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by more than four straight lines.

XXIV.

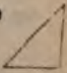
Of three-sided figures, an equilateral triangle is that which has three equal sides.

XXV.

An isosceles triangle is that which has only two sides equal.

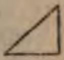
XXVI.

A scalene triangle is that which has three unequal sides.



XXVII.

A right-angled triangle is that which has a right angle.



XXVIII.

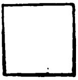
An obtuse-angled triangle is that which has an obtuse angle.

XXIX.

An acute-angled triangle is that which has three acute angles.

XXX.

Of four-sided figures, a square is that which has all its sides equal, and all its angles right angles.

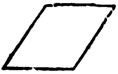


XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.

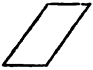
XXXII.

A rhombus is that which has all its sides equal, but its angles are not right angles.



XXXIII.

A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.



XXXIV.

All other four-sided figures, besides these are called Trapeziums.

XXXV.

Parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

POSTULATES.

I.

Let it be granted, that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

I.

Things which are equal to the same thing, are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same are equal to one another.

VII.

Things which are halves of the same are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot inclose a space.

XI.

All right angles are equal to one another

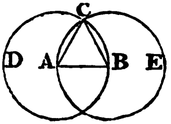
XII.

"If a straight line meets two straight lines, so as "to make the two interior angles on the same "side of it taken together less than two right "angles, these straight lines, being continually "produced, shall at length meet upon that side "on which are the angles which are less than "two *right angles.*"

PROP. I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given |; it is req^{d} to desc. an equilat. $\triangle$ on AB.



From cent. A, at dist. AB,
Postulate 3. desc. $\odot$ BCD; from cent. B, at dist. BA, desc. $\odot$ ACE; and from p^{t} C, in which these $\odot^{s}$ cut
Post. 1. one another, draw |^{s} CA, CB: ABC shall be an equilat. $\triangle$.

For, $\because$ A is cent. of $\odot$ BCD,
Definition 15. $\therefore$ AC = AB;
and, $\because$ B is cent. of $\odot$ ACE,
Def. 15. $\therefore$ BC = BA:
but, from above, AC = AB;
$\therefore$ AC, BC each = AB:
Axiom 1. $\therefore$ AC = BC;
$\therefore$ AC = BC = AB.

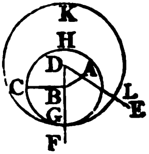
$\therefore$ *the triangle* ABC *is equilateral, and it is described on the straight line* AB. [Q. E. F.]

PROP. II. PROB.

From a given point to draw a straight line equal to a given straight line.

Let p^{t} A and | BC be given; it is req^{d} to draw from A a | = BC.

Draw $|$ AB, and upon it desc. the equilat. $\triangle$ DAB ; prod. the $|^s$ DA, DB to E, F ; from cent. B, at dist. BC, desc. $\odot$ CGH ; from cent. D, at dist. DG, desc. $\odot$ GKL : AL shall be $=$ BC. Post. 1. I. 1. Post. 2. Post. 3.



For, $\because$ B is cent. of $\odot$ CGH,
$\therefore$ BC $=$ BG ; Def. 15.
and, $\because$ D is cent. of $\odot$ GKL,
$\therefore$ DL $=$ DG ; Def. 15.
also, part DA $=$ part DB ; Constr.
$\therefore$ rem^r AL $=$ rem^r BG : Ax. 3.
but from above, BC $=$ BG ;
$\therefore$ AL, BC each $=$ BG :
$\therefore$ AL $=$ BC. Ax. 1.

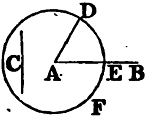
$\therefore$ *from the point* A, *is drawn a straight line equal to the given straight line* BC. [Q. E. F.]

PROP. III. PROB.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given $|^s$, of w^h AB is $>$ C : it is req^d to cut off from AB a part $=$ C. 2. 1.
From A draw $|$ AD $=$ C ; and from cent. A, at dist. AD, desc. $\odot$ DEF : AE shall be $=$ C. Post. 3.

For, $\because$ A is cent. of $\odot$ DEF,
$\therefore$ AE $=$ AD ; Def. 15
but C $=$ AD ; Constr.
$\therefore$ AE, C each $=$ AD ;
$\therefore$ AE $=$ C. Ax. 1.



$\therefore$ *from* AB *is cut off a part equal to* C.

[Q. E. F.]

PROP. IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another; they shall likewise have their bases, or third sides, *equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite.*

In the two $\triangle^s$ ABC, DEF, let the two sides AB, AC = the two DE, DF, each to each, viz.

AB = DE, AC = DF;

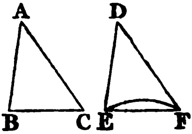
and also,

$\angle$ BAC = $\angle$ EDF;

then shall

base BC = base EF,

$\triangle$ ABC = $\triangle$ DEF;



and the rem^s $\angle^s$ = the rem^s $\angle^s$,

those to w^h the = sides are opp.

viz. $\angle$ ABC = $\angle$ DEF, and $\angle$ ACB = DFE.

For, let $\triangle$ ABC be applied to $\triangle$ DEF, so that p^t A may be on D, and side AB on DE: then

Hyp. $\because$ AB coincides with DE,

and also AB = DE,

$\therefore$ p^t B shall coincide with E:

Hyp. And $\because$ AB coincides with DE,

and also $\angle$ BAC = $\angle$ EDF,

$\therefore$ AC shall coincide with DF:

Hyp. but also AC = DF;

$\therefore$ p^t C shall coincide with F:

and it has been shown that

pt B coincides with E;

∴ base BC shall coincide with base EF;

for, pt B coinciding with E, and C with F, if BC do not coincide with EF, two |s will inclose a space:

but this is impossible. Ax. 10.

∴ base BC coincides with, and is = base EF; Ax. 8.

and

∴ △ ABC coincides with, and is = △ DEF;

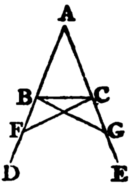
and the remg ∠s of the one △ coincide with and are = the remg ∠s of the other △, viz.

∠ ABC = ∠ DEF, ∠ ACB = ∠ DFE.

∴ *if two triangles have, &c.* [Q. E. D.]

PROP. V THEOR.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.



Let ABC be an isos. △, in wh side AB = side AC:

and let AB, AC be prodd to D, E:

then shall ∠ ABC = ∠ ACB,

and ∠ CBD = ∠ BCE.

In BD take any pt F; from AE, the >, cut off AG = AF, the <; 3. 1.

and join FC, GB.

Then, in △s AFC, AGB,

∵ { side AF = side AG, Constr
side AC = side AB, Hyp.
and ∠ FAG is com. to both;

4\. 1. $\therefore$ base FC = base GB, $\triangle$ AFC = $\triangle$ AGB, and
the rem^{g} $\angle^{s}$ of the one = the rem^{g} $\angle^{s}$ of the other, those to w^{h} the = sides are opp.
viz. $\angle$ ACF = $\angle$ ABG, and $\angle$ AFC = $\angle$ AGB.

Again,

Constr. $\because$ the whole AF = the whole AG,
Hyp. of w^{h}, the part AB = the part AC,
Ax. 3. $\therefore$ the rem^{r} BF = the rem^{r} CG:
and, from above, FC = GB:

Hence, in $\triangle^{s}$ BFC, CGB,

$\because \left\{ \begin{array}{l} \text{side BF} = \text{CG, FC} = \text{GB,} \\ \text{and } \angle \text{ BFC} = \angle \text{ CGB,} \end{array} \right.$

4\. 1. $\therefore$ $\triangle$ BFC = $\triangle$ CGB,
and the $\angle^{s}$ of the one = the $\angle^{s}$ of the other,
viz. $\angle$ FBC, = $\angle$ GCB, $\angle$ BCF = $\angle$ CBG:
and since it has been shown that
the whole $\angle$ ABG = the whole $\angle$ ACF,
and also,
the part CBG = the part BCF,
Ax. 3. $\therefore$ the rem^{g} $\angle$ ABC = the rem^{g} $\angle$ ACB:
and these are the $\angle^{s}$ at the base of $\triangle$ ABC.
It has also been proved that
$\angle$ FBC = $\angle$ GCB;
w^{h} are the $\angle^{s}$ on the other side of the base.
$\therefore$ *the angles at the base, &c.*

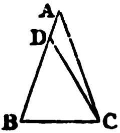
[Q. E D.]

Cor.—Hence every equilat. $\triangle$ is also equiang

PROP. VI. THEOR.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to *the equal angles shall be equal to one another.*

In $\triangle$ ABC, let $\angle$ ABC $=$ $\angle$ ACB:
then shall the side AB $=$ side AC.



For, if AB be $\neq$ AC, one is $>$ the other: let AB be the $>$; from it cut off DB $=$ AC; and join DC. 3. 1.

Then, in $\triangle^s$ DBC, ACB,

$\because$ { side DB $=$ AC, BC is com. to both, Constr.
and also, $\angle$ DBC $=$ $\angle$ ACB; Hyp.

$\therefore$ the base DC $=$ the base AB, 4. 1.
and $\triangle$ DBC $=$ $\triangle$ ACB,
i.e. the $<$ $=$ the $>$,
wh is absurd.
$\therefore$ AB is not $\neq$ AC,
i.e. AB $=$ AC.

$\therefore$ *if two angles, &c.* [Q. E. D.]

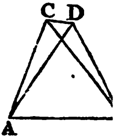
COR.—Hence every equiang. $\triangle$ is also equilat.

PROP. VII. THEOR.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides, which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

If possible, on the same base AB, and on the same side of it, let there be two $\triangle^s$ ACB, ADB,

such that their sides CA, DA, terminated in the extr^y A of the base, shall be = one another, and likewise those CB DB, that are terminated in B.



Join CD: and first let the vertex of each △ be without the other △; the

Hyp. $\because \quad AC = AD,$

5. 1. $\therefore \angle ACD = \angle ADC:$

Ax. 9. but $\angle ACD > \angle BCD;$

$\therefore \angle ADC > \angle BCD;$

à fortiori, $\therefore \angle BDC > \angle BCD.$

On the other hand,

Hyp. $\because \quad BC = BD,$

5. 1. $\therefore \angle BDC = \angle BCD:$

but, from above,

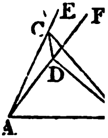
$\angle BDC > \angle BCD,$

i.e. $\angle$ BDC is both > and = the same $\angle$ BC

w^h is impossible.

If, next, one of the vertices, as D, be within other △, prod. AC, AD to E, F:

then, in △ ACD,



Hyp. $\because$ side AC = side AD,

5. 1. $\therefore \angle ECD = \angle FDC:$

Ax. 9. but $\angle ECD > \angle BCD;$

$\therefore \angle FDC > \angle BCD;$

à fortiori, $\therefore \angle BDC > \angle BCD.$

On the other hand,

Hyp. $\because$ side BD = side BC,

5. 1. $\therefore \angle BDC = \angle BCD:$

but, from above,

$\angle BDC > \angle BCD,$

i.e. $\angle$ BDC is both $>$ and $=$ the same $\angle$ BCD, w[h] is impossible.

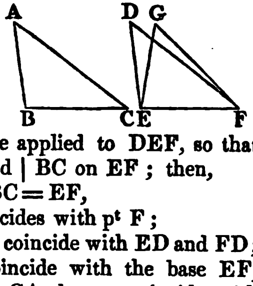
The case in which the vertex of one $\triangle$ is on a side of the other needs no demonstration.

$\therefore$ *on the same base, &c.* [Q. E. D.]

PROP. VIII. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

In the two $\triangle$s ABC, DEF, let the two sides AB, AC $=$ the two DE, DF, each to each, viz. AB$=$DE, AC$=$DF; and also, let the base BC $=$ the base EF: then shall $\angle$ BAC $= \angle$ EDF.



For, let $\triangle$ ABC be applied to DEF, so that p[t] B may be on E, and | BC on EF; then,

$\because$ BC $=$ EF, Hyp.

$\therefore$ p[t] C coincides with p[t] F;

$\therefore$ BA and CA shall coincide with ED and FD; for, if the base BC coincide with the base EF, whilst the sides BA, CA do not coincide with those ED, FD, but have a different situation, as EG, FG; then, on the same base EF, and on

the same side of it, there can be two $\triangle^s$ such that their sides w[h] are terminated in one extr[y] of the base, are = one another, and likewise those w[h] are terminated in the other extr[y]:

7. 1. but this is impossible:

$\therefore$ if the base BC coincide with EF, the sides BA, CA cannot but coincide with those ED, FD;

$\therefore$ $\angle$ BAC must also coincide with $\angle$ EDF,

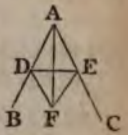
Ax. 8. and $\therefore$ $\angle \text{BAC} = \angle \text{EDF}$.

$\therefore$ *if two triangles, &c.* [Q. E. D.]

PROP. IX. PROB.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rect[l] $\angle$; it is req[d] to bis[t] it.



3. 1. Take any p[t] D in AB; from AC cut off AE = AD; join DE; on it

1. 1. desc. the equilat. $\triangle$ DEF; and join AF: $\angle$ BAC shall be bis[d] by | AF.

For, in $\triangle^s$ DAF, EAF,

Constr. $\because \left\{ \begin{array}{l} \text{side AD} = \text{AE}, \\ \text{AF is com. to both,} \\ \text{and base DF} = \text{base EF}; \end{array} \right.$

8. 1. $\therefore$ $\angle \text{DAF} = \angle \text{EAF}$,

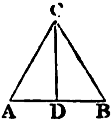
$\therefore$ *the given angle* BAC *is bisected by the straight line* AF.

[Q. E. F.]

PROP. X. PROB.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given | ; it is req[d] to bis[t] it.



Desc. on AB the equilat. $\triangle$ ABC, and bis[t] the $\angle$ ACB by | CD ; AB shall be bis[d] in p[t] D. 1. 1. 9. 1.

For, in $\triangle$[s] ACD, BCD,

$\because$ { side AC = side BC, Constr.
CD is common to both,
and $\angle$ ACD = $\angle$ BCD ; }

$\therefore$ base AD = base DB : 4. 1.

And $\therefore$ *the straight line* AB *is bisected in point* D.

[Q. E. F.]

PROP. XI. PROB.

To draw a straight line at right angles to a given straight line, from a given point in the same.

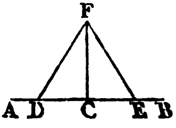
In the given | AB let the p[t] C be given ; it is req[d] to draw from C a | at r[t] $\angle$[s] to AB.

Take any p[t] D in AC, and make CE = CD ; 3. 1.
on DE desc. the equilat. $\triangle$ DEF, and join CF :
CF, drawn from the p[t] C, shall be at r[t] $\angle$[s] to | AB. 1. 1.

For, in $\triangle$[s] DCF, ECF,

$\because$ { side DC = side EC, Constr
FC is common to both,
and base DF = base EF ; }

$\therefore$ $\angle$ DCF = $\angle$ ECF ; 8. 1.
and they are *adj*[t] $\angle$[s]:



but when the adj^t $\angle^s$, w^h one | makes with another,
ef. 10. are $=$ one another, each is called a r^t $\angle$:
$\therefore$ each of the $\angle^s$ DCF, ECF is a r^t $\angle$.

And $\therefore$ *from the given point* C, *in the given straight line* AB, *has been drawn a straight line* FC *at right angles to* AB.

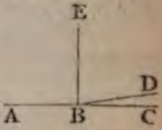
[Q. E. F.]

Cor. — Hence it may be shown that two |^s cannot have a com. seg^t.

For if it be possible, let the two |^s ABC, ABD have the com. seg^t AB.

From p^t B draw BE at r^t $\angle^s$ to AB; then,

$\because$ ABC is a |,
ef. 10. $\therefore \angle$ CBE $= \angle$ EBA;
and $\because$ ABD is a |,
$\therefore \angle$ DBE $= \angle$ EBA;
x. 1. $\therefore \angle$ DBE $= \angle$ CBE,
i. e. the $<\,=$ the $>$,
but this is impossible.

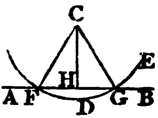


$\therefore$ *two straight lines cannot have a common segment.*

PROP. XII. PROB.

To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.

Let AB be the given |, and C the given p^t without it: it is req^d to draw from C a $\perp$ to AB.



Take any p^t D on the other side of AB; from cent. C, at dist. CD, desc. ⊙ EGF, meeting AB in F and G; bis^t FG in H and join CH: CH shall be a $\perp$ to AB. Post. 3. 10. 1.

Join CF, CG; then in $\triangle^s$ FHC, GHC,

$\because \begin{cases} \text{side FH} = \text{side HG}, \\ \text{HC is common to both}, \\ \text{and base CF} = \text{base CG}; \end{cases}$ Constr Def. 15.

$\therefore$ $\angle$ CHF $=$ $\angle$ CHG; 8. 1.

and these are adj^t $\angle^s$:

but when one |, standing on another |, makes the adj^t $\angle^s$ = one another, each of these $\angle^s$ is a r^t $\angle$; and the | w^h stands on the other is called a $\perp$ to it. Def. 10.

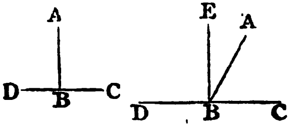
$\therefore$ *from the given point* C *has been drawn a perpendicular* CH *to the given straight line* AB.

[Q. E. F.]

PROP. XIII. THEOR.

The angles which one straight line makes with another, upon one side of it, are either two right angles, or are together equal to two right angles.

Let | AB make with CD, on one side of it, the $\angle^s$ CBA, DBA: these shall either be two r^t $\angle^s$, or shall together be = two r^t $\angle^s$.



For, if $\angle$ CBA $=$ $\angle$ DBA, Def 10

each of them is a r^t $\angle$:

But if $\angle$ CBA $\neq \angle$ DBA,

11. 1. from p^t B draw BE at right $\angle^s$ to CD;

Def. 10. $\therefore$ each of the $\angle^s$ CBE, DBE will be a $r^t \angle$.

Now, $\angle$ CBE $= \angle^s$(ABC+ABE);

let $\angle$ DBE be added:

then,

Ax. 2. $\angle^s$(CBE+DBE) $= \angle^s$(ABC+ABE+DBE).

Again, $\angle$ DBA $= \angle^s$(DBE+ABE)

let $\angle$ ABC be added:

then,

Ax. 2. $\angle^s$(DBA+ABC) $= \angle^s$(DBE+ABE+ABC):

but,

$\angle^s$(CBE+DBE) = these same three $\angle^s$;

Ax. 1. $\therefore \angle^s$(CBE+DBE) $= \angle^s$(DBA+ABC):

but CBE, DBE are two $r^t \angle^s$;

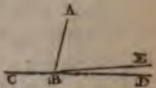
Ax. 1. $\therefore \angle^s$(DBA+ABC) = two $r^t \angle^s$.

$\therefore$ *the angles, &c.* [Q.E.D.]

PROP. XIV. THEOR.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the p^t B in | AB, let the two $|^s$ BC, BD, on the opp. sides of AB, make the adj^t $\angle^s$ (ABC+ABD) = two $r^t \angle^s$: BC, BD shall be in the same |.



For

if BD be not in the same | with BC,

Post. 2. *let* BE be in the same | with it:

then,

$\because$ | AB makes with | CBE, on one side of it, the $\angle^s$ ABC, ABE,

$\therefore$ $\angle^s$(ABC+ABE)=two r$^t\angle^s$: 13. 1.

but $\angle^s$(ABC+ABD)=two r$^t\angle^s$; Hyp.

$\therefore$ $\angle^s$(ABC+ABE)=$\angle^s$(ABC+ABD): Ax. 1.

let the com. $\angle$ ABC be taken away;

then, the rem^g $\angle$ ABE=rem^g $\angle$ ABD, Ax. 3.

i. e. the <=the >,

w^h is impossible:

$\therefore$ BE is not in the same | with BC.

And it may in like manner be shown that no other can be in the same | with it but BD;

$\therefore$ BD is in the same | with CB.

$\therefore$ *if at a point, &c.* [Q. E. D.]

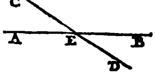
PROP. XV. THEOR.

If two straight lines cut one another, the vertical, or opposite, *angles shall be equal.*

Let the two |^s AB, CD, cut one another in E:

then, $\angle$ AEC=opp. $\angle$ DEB,

and $\angle$ CEB=opp. $\angle$ AED.



For,

$\because$ | AE makes with CD the $\angle^s$ CEA, AED, 13. 1.

$\therefore$ $\angle^s$(CEA+AED)=two r$^t\angle^s$:

And,

$\because$ | DE makes with AB the $\angle^s$ AED, DEB,

$\therefore$ also $\angle^s$(AED+DEB)=two r$^t\angle^s$; 13. 1.

$\therefore$ $\angle^s$(CEA+AED)=$\angle^s$(AED+DEB). Ax. 1.

Ax. 3. Let the com. ∠ AED be taken away;
then, rem^g ∠ CEA=rem^g ∠ DEB:
and in the same manner it may be shown that
∠ CEB=∠ AED.

∴ *if two straight lines, &c.* [Q.E.D.]

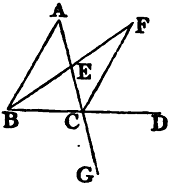
Cor. 1.—Hence it is manifest that, if two |^s cut
13. 1. one another, the ∠^s w^h they make at the p^t where
they cut, are together=four r^t ∠^s.

Cor. 2.—And consequently, all the ∠^s made by any number of |^s meeting in one p^t are together =four r^t ∠^s.

PROP. XVI. THEOR.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let the side BC of the △ABC be prod^d to D; the ext^r ∠ ACD shall be > either of the int^r and opp. ∠^s CBA, BAC.



10. 1. Bis^t AC in E; join BE;
prod. BE to F, making EF
3. 1. =BE, and join FC.

Then, in △^s AEB, CEF,

Constr. ∵ side AE=EC,
BE=EF,
15. 1. and ∠ AEB=opp. ∠ CEF,

∴ the base AB=the base CF,
4. 1. △AEB=△CEF,
and the rem^g ∠^s=the rem^g ∠^s:

$\therefore \angle BAE = \angle ECF$;
but $\angle ECD > \angle ECF$;
$\therefore \angle ACD > \angle BAC$:

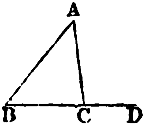
And in like manner, if the side BC be bis[d], and AC prod[d] to G, it may be shown that
$\angle BCG$, *i.e.* $\angle ACD > ABC$ 15. 1.

$\therefore$ *if one side, &c.* [Q. E. D.]

PROP. XVII. THEOR.

Any two angles of a triangle are together less than two right angles.

Let ABC be any $\triangle$: any two of its $\angle^s$ are together $<$ two $r^t \angle^s$



Prod. BC to D; then
ext$^r \angle ACD >$ int$^r \angle ABC$: 16. 1.
let $\angle ACB$ be added:
then, $\angle^s(ACD+ACB) > \angle^s(ABC+ACB)$:
but, $\angle^s(ACD+ACB) =$ two $r^t \angle^s$; 13. 1.
$\therefore \angle^s(ABC+ACB) <$ two $r^t \angle^s$:
and in like manner it may be shown that
$\angle^s(BAC+ACB) <$ two $r^t \angle^s$,
$\angle^s(BAC+ABC) <$ two $r^t \angle^s$.

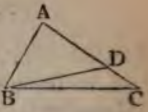
$\therefore$ *any two angles, &c.* [Q. E. D.]

PROP. XVIII. THEOR.

The greater side of every triangle is opposite to the greater angle.

In any $\triangle$ ABC, let side AC be $>$ side AB; $\angle ABC$ shall be $> \angle ACB$.

3. 1. Make $AD = AB$, and join BD;
then, of $\triangle BDC$,



16. 1. ext^r $\angle ADB >$ int^r DCB;
but $\because$ side $AD =$ side AB,
5. 1. $\therefore \angle ADB = \angle ABD$;
$\therefore \angle ABD > \angle ACB$;
à fortiori,
$\therefore \angle ABC > \angle ACB$.

$\therefore$ *the greater side, &c.* [Q.E.D.]

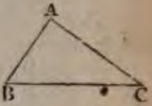
PROP. XIX. THEOR.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

In any $\triangle ABC$, let $\angle ABC$ be $> \angle ACB$: side AC shall be $>$ side AB.

For

AC must be $>, =,$ or $< AB$.



Now, if $AC = AB$,
5. 1. then must $\angle ABC = \angle ACB$:
but this is not the case;
$\therefore AC$ is $\neq AB$.
Next, if AC be $< AB$,
18. 1. then must $\angle ABC$ be $< ACB$:
but this is not the case;
$\therefore$ AC is $\nless AB$:
neither is $AC = AB$:
$\therefore$ AC must be $> AB$.

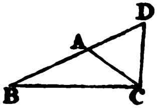
$\therefore$ *the greater angle &c.* [Q.E.D.]

PROP. XX. THEOR.

Any two sides of a triangle are together greater than the third side.

Let ABC be a $\triangle$: any two of its sides are together $>$ the third; viz.

$$(AB+AC)>BC,$$
$$(AB+BC)>AC,$$
$$(BC+CA)>AB.$$



Prod. BA to D, making AD=AC, and join DC. 3. 1.

Then, $\because$ AD=AC,

$\therefore \angle$ ADC$=\angle$ACD: 5. 1

but $\angle$ BCD$>\angle$ACD;

$\therefore \angle$ BCD$>\angle$ADC

or $\angle$BDC:

and,

$\because$ the $>\angle$ of a $\triangle$ is subtended by the $>$ side, 19. 1.

$\therefore$ side BD $>$ side BC:

but BD$=$(BA$+$AC); Constr.

$\therefore$(BA$+$AC)$>$BC:

and in like manner it may be shown that

$$(AB+BC)>AC,$$
$$(BC+AC)>AB.$$

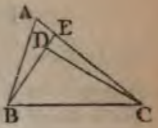
$\therefore$ *any two sides, &c.* [Q. E. D.]

PROP. XXI. THEOR.

If, from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a $\triangle$; and from B, C, the ends of

a side BC, let two |s BD, CD be drawn to a p^t D within the △. then shall (BD+CD) be <(AB+AC), but ∠BDC be > ∠BAC.



Prod. BD to E; then,

20. 1. ∵ any two sides of a △ are > the third,

∴ (AB+AE)>BE:

let EC be added; then

(AB+AC)>(BE+EC)

Again, in △CED,

20. 1. (CE+ED)>CD:

let DB be added; then

(CE+EB)>(CD+DB):

but, from above,

(AB+AC)>(BE+EC);

à fortiori, ∴ (AB+AC)>(CD+DB)

Again,

16. 1. ∵ the ext[r] ∠ of a △ is > the int[r] and opp. ∠,

∴ ∠BDC>CED;

and, for the same reason,

∠CED>BAC:

but, from above, ∠BDC>CED;

à fortiori, ∴ ∠BDC>BAC.

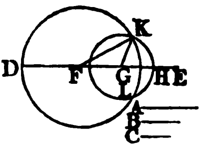
∴ *if from the ends of, &c.* [Q.E.D.]

PROP. XXII. PROB.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let A, B, C be three given |s of w[h] any two are > the third, viz.

$(A+B)>C$,
$(A+C)>B$,
$(B+C)>A$:
it is req[d] to form a $\triangle$, the sides of w[h] shall be severally $=$ A, B, C.



Take a | DE, terminated at D, but unlimited towards E, and make $DF=A$, $FG=B$, $GH=C$; 3. 1.
then, from cent. F, at dist. FD, desc. $\odot$ DKL; Post. 3.
from cent. G. at dist. GH. desc. another $\odot$ HLK, and join KF, KG: $\triangle$ KFG shall have its sides severally $=$ the three |s A, B, C.

For, $\because$ F is cent. of $\odot$ DKL,
$\therefore$ $FD=FK$: Def. 15.
but $FD=A$; Constr.
$\therefore$ $FK=A$.
Again, $\because$ G is cent. of $\odot$ HKL,
$\therefore$ $GH=GK$; Def. 15.
but $GH=C$; Constr.
$\therefore$ $GK=C$:
and $FG=B$;
$\therefore$ the |s FK, FG, GK $=$ the three A, B, C.

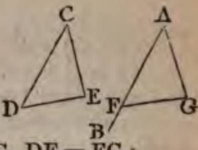
And $\therefore$ *the triangle* KFG *has its sides equal to the three given straight lines* A, B, C.

[Q. E. F.]

PROP. XXIII. PROB.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given |, A the given p[t] in it, and DCE the given $\angle$: it is req[d] to make at A in | AB an $\angle$ that shall be $=$ DCE.



In CD, CE take any p[ts] D, E, and join DE; then make the $\triangle$ AFG, the
22. 1. sides of w[h] shall be = the three $|$[s] CD, DE, EC, viz.

CD = AF, CE = AG, DE = FG;

then shall $\angle$ FAG = $\angle$ DCE.

For, in $\triangle$[s] DCE, FAG,

$$\because \begin{cases} \text{side DC} = \text{FA, CE} = \text{AG}, \\ \text{and base DE} = \text{base FG}; \end{cases}$$

8. 1. $\therefore$ $\angle$ DCE = $\angle$ FAG.

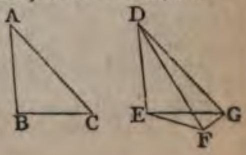
And $\therefore$ *at the given point* A *in the given straight line* AB, *the angle* FAG *is made equal to the given angle* DCE. [Q. E. F.]

PROP. XXIV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each; but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let the two $\triangle$[s] ABC, DEF, have the sides AB = DE, and AC = DF; but the $\angle$ BAC > EDF: the base BC shall be > the base EF.

Of the two sides DE, DF, let DE be that w[h] is $\not>$ the other; at the
23. 1. p[t] D, in $|$ DE, make $\angle$ EDG = $\angle$ BAC;
3. 1. also, make DG = AC or DF, and join EG, GF



Then, in $\triangle^s$ ABC, DEG,

$\because \begin{cases} \text{side } AB = DE,\ AC = DG, \\ \text{and } \angle\, BAC = EDG, \end{cases}$

$\therefore$ the base $BC =$ the base EG. 4. 1.

And, $\because$ $DG = DF$,

$\therefore$ $\angle\, DGF = \angle\, DFG$: 5. 1.

but $\angle\, DGF > \angle\, EGF$;

$\therefore$ $\angle\, DFG > \angle\, EGF$;

à fortiori, $\therefore$ $\angle\, EFG > \angle\, EGF$:

But the $> \angle$ is subtended by the $>$ side; 19. 1

and $\therefore$ the side $EG >$ the side EF.

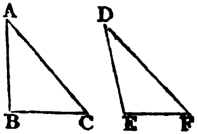
but, from above, $EG = BC$;

and $\therefore$ $BC > EF$.

$\therefore$ *if two triangles, &c.* [Q. E. D.]

PROP. XXV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each; but the base of the one greater than the base of the other; the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.



Let the two $\triangle^s$ ABC, DEF, have the sides $AB = DE$, $AC = DF$; but the base $BC > EF$: $\angle\, BAC$ shall be $> \angle\, EDF$.

For,

$\angle\, BAC$ must be $>$, $=$ or $<$ EDF

Now, if $\angle$ BAC = EDF,

4.1. then must base BC = EF;

but this is not the case;

$\therefore \angle$ BAC $\neq$ EDF.

Again, if $\angle$ BAC < EDF,

24.1. then also base BC < EF:

but this is not the case;

$\therefore \angle$ BAC $\not<$ EDF;

also, $\angle$ BAC $\neq$ EDF;

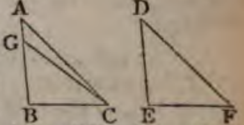
and $\therefore \angle$ BAC > EDF.

$\therefore$ *if two triangles, &c.* [Q. E. D.]

PROP. XXVI. THEOR.

If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, viz., either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other.

In two $\triangle$s ABC, DEF, let $\angle$ ABC = $\angle$ DEF, $\angle$ ACB = $\angle$ DFE; also, one side = one side: and, first, let those sides be = w[h] are adj[t] to the $\angle$s that are = in the two $\triangle$s, viz. BC = EF; the other sides shall be =, each to each, viz. side AB = DE, AC = DF; and also, the third $\angle$ BAC = the third $\angle$ EDF.



For, if AB $\neq$ DE, one must be > the other;

3.1. let AB be > DE; make BG = DE, and join GC;

Then, in $\triangle^s$ GBC, DEF,

$\because \begin{cases} \text{side } BG = DE,\ BC = EF, \\ \text{and } \angle\ GBC = \angle\ DEF, \end{cases}$ Constr.

$\therefore \begin{cases} \text{the base } GC = \text{the base } DF, \\ \text{the } \triangle\ GBC = \text{the } \triangle\ DEF, \\ \text{and the rem}^{g}\ \angle^{s} = \text{the rem}^{s}\ \angle^{s}, \\ \text{each to each:} \end{cases}$ 4. 1.

$\therefore\ \angle\ BCG = \angle\ DFE$:

but $\angle\ DFE = \angle\ BCA$ Hyp.

$\therefore\ \angle\ BCG = \angle\ BCA$,

or the $<$ = the $>$,

w^h is impossible.

$\therefore$ AB is not $\neq$ DE,

i.e. AB = DE:

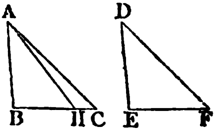
Hence, in $\triangle^s$ ABC, DEF,

$\because \begin{cases} \text{side } AB = DE,\ BC = EF, \\ \text{and } \angle\ ABC = DEF; \end{cases}$

and $\therefore \begin{cases} \text{the base } AC = \text{the base } DF, \\ \text{and } \angle\ BAC = \angle\ EDF. \end{cases}$ 4. 1.

Next, let the sides w^h are opp. to the = $\angle^s$ in each $\triangle$ be = one another, viz. AB = DE; in this case also the other sides shall be =, viz. AC=DF, BC=EF; and also $\angle$ BAC = $\angle$ EDF.

For, if BC $\neq$ EF,
let BC be $>$ EF;
make BH=EF,
and join AH.



Then, in $\triangle^s$ ABH, DEF,

$\because \begin{cases} \text{side } BH = EF,\ AB = DE, \\ \text{and } \angle\ ABH = \angle\ DEF, \end{cases}$

$$\therefore \left\{ \begin{array}{c} \text{the base } AH = \text{the base } DF \\ \triangle ABH = \triangle DEF, \\ \text{and the rem}^{g} \angle^{s} = \text{the rem}^{g} \angle^{s}, \\ \text{each to each.} \end{array} \right.$$ 4. 1

$\therefore \angle BHA = \angle EFD$:

Hyp. but $\angle EFD = \angle BCA$;

$\therefore \angle BHA = \angle BCA$,

or, the ext^{r} $\angle$ of a $\triangle$ = the int^{r} and opp. $\angle$;

16. 1. but this is impossible.

$\therefore BC$ is not $\neq EF$,

i. e. $BC = EF$,

Hence, in $\triangle^{s}$ ABC, DEF,

$$\left\{ \begin{array}{l} \text{side } AB = DE,\ BC = EF, \\ \text{and } \angle ABC = \angle DEF, \end{array} \right.$$

4. 1. and $\therefore$ $\left\{ \begin{array}{l} \text{the base } AC = \text{the base } DF \\ \text{and } \angle BAC = \angle EDF. \end{array} \right.$

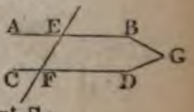
$\therefore$ *if two triangles, &c.* [Q. E. D.]

PROP. XXVII. THEOR.

If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel.

Let the | EF, w^{h} falls on the two |^{s} AB, CD make the alt. $\angle^{s}$ AEF, EFD = one another:

AB shall be $\parallel$ CD.

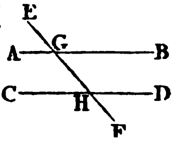


For, if not, AB and CD, Def. 35. being prod^{d}, will meet either towards B, D, or towards A, C: let them be so prod^{d} and meet, *if possible*, towards B, D, in p^{t} G:

Then, ∵ GEF is a △,
∴ $ext^r \angle AEF > int^r$ and opp. $\angle EFG$: 16. 1.
but, also $\angle AEF = \angle EFG$; Hyp.
w^h is impossible.
∴ AB, CD, being $prod^d$, do not meet towards B, D.
And in like manner it may be shown that they do not meet towards A, C.
But those $|^s$ w^h, though $prod^d$ ever so far, meet neither way, are ∥ one another ; Def. 35
∴ AB is ∥ CD.
∴ *if a straight line, &c.* [Q. E. D.]

PROP. XXVIII. THEOR.

If a straight line falling upon two other straight lines makes the exterior angle equal to the interior and opposite upon the same side of the line, or makes the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.



Let the | EF, w^h falls on the two $|^s$ AB, CD, make the ext^r $\angle EGB$ = the int^r and opp. $\angle GHD$ on the same side; or make the $int^r \angle^s (BGH + GHD)$ = two $r^t \angle^s$: AB shall be ∥ CD.

For, ∵ $\angle EGB = \angle GHD$, Hyp.
and $\angle EGB = \angle AGH$, 15. 1.
∴ $\angle AGH = \angle GHD$: Ax. 1.
and they are the alt. $\angle^s$;
∴ AB is ∥ CD 27. 1.

Again,

Hyp. $\because \angle^s(BGH + GHD) =$ two rt $\angle^s$,

and also,

13. 1. $\angle^s(AGH + BGH) =$ two rt $\angle^s$,

$\therefore \angle^s(AGH + BGH) = \angle^s(BGH + GHD$

take away the com. $\angle$ BGH;

Ax. 3. then, the remg $\angle$ AGH = remg $\angle$ GHD:

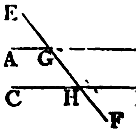
and they are alt. $\angle^s$:

27. 1. $\therefore$ AB is $\parallel$ CD.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. XXIX. THEOR.

If a straight line fall upon two parallel straig lines, it makes the alternate angles equal to o another; and the exterior angle equal to the i terior and opposite upon the same side; a likewise the two interior angles upon the sa side together equal to two right angles.



Let the | EF fall on the $\parallel^s$ AB, CD: then shall

$\angle$ AGH = the alt. $\angle$ GHD,

$\angle$ EGB = the intr $\angle$ GHD,

and also the two intr $\angle^s$

(BGH + GHD) = two rt $\angle^s$.

For if $\angle$ AGH be $\neq \angle$ GHD,

one must be $>$ the other:

let AGH be the greater of the two $\angle^s$,

and add $\angle$ BGH to each;

then

$\angle^s(AGH + BGH) > \angle^s(BGH + GHD)$

but

$\angle^s(\text{AGH}+\text{BGH}) = \text{two r}^t \angle^s$; 13. 1.

$\therefore \angle^s(\text{BGH}+\text{GHD}) < \text{two r}^t \angle^s$: Ax. 1.

but those $|^s$ w^h, with another $|$ falling on them, make the int^r $\angle^s$ on the same side $<$ two r^t $\angle^s$, will meet if continually $prod^d$; Ax. 12.

$\therefore$ $|^s$ AB, CD, if $prod^d$ far enough, will meet:

but these $|^s$ are $\parallel$,

and $\therefore$ never meet: Def. 35.

$\therefore \angle$ AGH is not $\neq$ GHD,

i.e. $\angle \text{AGH} = \text{GHD}$:

but $\angle \text{AGH} = \text{EGB}$, 15. 1.

$\therefore$ $\angle \text{EGB} = \text{GHD}$:

add to each $\angle$ BGH;

then,

$\angle^s(\text{EGB}+\text{BGH}) = \angle^s(\text{GHD}+\text{BGH})$:

but $\angle^s(\text{EGB}+\text{BGH}) = \text{two r}^t \angle^s$; 13. 1.

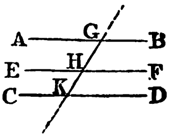
$\therefore \angle^s(\text{GHD}+\text{BGH}) = \text{two r}^t \angle^s$. Ax. 1.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. XXX. THEOR.

Straight lines which are parallel to the same straight line are parallel to each other.

Let AB, CD be each $\parallel$ EF:
AB shall be $\parallel$ CD.



Let $|$ GHK cut the $|^s$ AB, EF and CD.

Then,

$\because$ GHK cuts the $\parallel^s$ AB, EF,

$\therefore$ $\angle \text{AGH} = \angle \text{GHF}$. 29. 1.

Again, ∵ GK cuts the $\|^s$ EF, CD,

29. 1. ∴ $\angle$ GHF $= \angle$ GKD:

but, from above,

$\angle$ AGK $= \angle$ GHF;

Ax. 1. ∴ $\angle$ AGK $= \angle$ GKD:

and they are alt. $\angle^s$;

27. 1. ∴ AB is $\|$ CD.

∴ *if a straight line, &c.* [Q. E. D.]

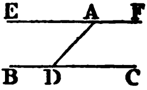
PROP. XXXI. PROB.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given p^t, and BC the given |; it is req^d to draw through A a $\|$ to BC.

In BC take any p^t D, join AD, at p^t A, in | AD, make $\angle$ DAE

23. 1. $= \angle$ ADC, and prod. | EA to F: EF shall be $\|$ BC.



For, ∵ | AD meets the two $|^s$ BC, EF, and makes $\angle$ EAD $=$ the alt. $\angle$ ADC

27. 1. ∴ EF is $\|$ BC.

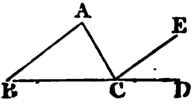
∴ *the straight line* EAF *is drawn through the given point* A, *and is parallel to the given straight line* BC. [Q. E. F.]

PROP. XXXII. THEOR.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite

angles; and the three interior angles of every triangle are equal to two right angles.

Let a side BC of any $\triangle$ ABC be prod[d] to D: the ext[r] $\angle$ ACD shall be = the two int[r] and opp. $\angle^s$(CAB + ABC); and the three int[r] $\angle^s$ of the $\triangle$, viz.



$\angle^s$(ABC + ACB + CAB) = two r[t] $\angle^s$.

Draw CE $\parallel$ AB: 31. 1.

then, $\because$ AB is $\parallel$ CE, and AC meets them,

$\therefore$ $\angle$ BAC = alt. $\angle$ ACE. 29. 1.

Again,

$\because$ AB is $\parallel$ CE, and BD falls upon them,

$\therefore$ ext[r] $\angle$ ECD = int[r] and opp. $\angle$ ABC: 29. 1.

and, from above,

$\angle$ ACE = $\angle$ BAC;

$\therefore$ the ext[r] $\angle$ ACD = $\left\{\begin{array}{l}\text{the two int}^r\text{ and opp.}\\ \angle^s(\text{BAC}+\text{ABC}):\end{array}\right.$ Ax. 2

add $\angle$ ACB; then,

$\angle^s$(ACD + ACB) = $\angle^s$(BAC + ABC + ACB) Ax 2

but

$\angle^s$(ACD + ACB) = two r[t] $\angle^s$: 13. 1

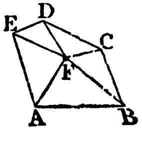
$\therefore$ $\angle^s$(BAC + ABC + ACB) = two r[t] $\angle^s$. Ax. 1

$\therefore$ *if a side, &c.* [Q. E. D.]

Cor. 1.

$\left.\begin{array}{l}\text{All the int}^r\ \angle^s\text{ of any}\\ \text{rect}^l\text{ fig.} + \text{four r}^t\ \angle^s\end{array}\right\} = \left\{\begin{array}{l}\text{twice as many r}^t\ \angle^s\text{ as}\\ \text{the fig. has sides.}\end{array}\right.$

For, any rect[l] fig. ABCDE can, by drawing $|^s$ from a p[t] F within the fig. to each $\angle$, be div[d] into as many $\triangle^s$ as the fig. has sides.



And by the prop^n,

the $\angle^s$ of each $\triangle$ = two r^t $\angle^s$;

$\therefore$ all the $\angle^s$ of the $\triangle^s$ = $\left\{\begin{array}{l}\text{twice as man} \\ \text{as there are} \\ \text{as there are} \\ \text{the fig.}\end{array}\right.$

But,

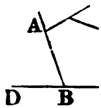
the same $\angle^s$ = the $\angle^s$ of the fig. + the $\angle$

Cor. 2. and the $\angle^s$ at F = four r^t $\angle^s$:

15. 1. $\therefore$ $\left.\begin{array}{l}\text{the } \angle^s \text{ of the fig.} \\ + \text{ four r}^t\ \angle^s\end{array}\right\} = \left\{\begin{array}{l}\text{twice as many} \\ \text{the fig. has}\end{array}\right.$

Cor. 2.—All the ext^r $\angle^s$ of any rect^l fig. t = four r^t $\angle^s$.

For,



13. 1. $\because$ $\left.\begin{array}{l}\text{each int}^r\ \angle \\ \text{ABC} \\ + \text{its adj}^t \text{ ext}^r \\ \angle \text{ ABD,}\end{array}\right\}$ = two r^t $\angle^s$,

$\therefore$ $\left.\begin{array}{l}\text{all the int}^r\ \angle^s \\ + \text{ all the ext}^r\ \angle^s\end{array}\right\} = \left\{\begin{array}{l}\text{twice as many r} \\ \text{there are } \angle^s \text{ c}\end{array}\right.$

Cor. 1. *i.e.* = the int^r $\angle^s$ + fou

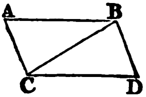
take away the com. int^r $\angle^s$;

then, all the ext^r $\angle^s$ = four r^t $\angle^s$.

PROP. XXXIII. THEOR.

The straight lines which join the extremitie equal and parallel straight lines towards t parts, are also themselves equal and par

Let | AB be = and $\parallel$ CD, and let the joined towards the same parts by the |^s A(AC shall be = and $\parallel$ BD.



Join BC: then,

$\because$ BC meets the $\parallel^s$ AB, CD,

$\therefore \angle$ ABC = alt. $\angle$ BCD; 29. 1.

Hence, in $\triangle^s$ ABC, DCB,

$\because \begin{cases} \text{side AB} = \text{CD, BC is com. to both,} \\ \text{and also, } \angle \text{ ABC} = \angle \text{ BCD,} \end{cases}$

$\therefore \begin{cases} \text{the base AC} = \text{the base BD,} \\ \triangle \text{ ABC} = \triangle \text{ BCD,} \\ \text{and the rem}^{g} \angle^{s} = \text{the rem}^{g} \angle^{s}, \\ \text{each to each:} \end{cases}$ 4. 1.

$\therefore \angle$ ACB = $\angle$ CBD:

and $\because$ | BC, w^h meets the two |^s AC, BD, makes

$\angle$ ACB = alt. $\angle$ CBD,

$\therefore$ AC is $\parallel$ BD: 27. 1.

and, from above, AC = BD.

$\therefore$ *the straight lines, &c.* [Q. E. D.]

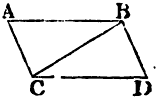
PROP. XXXIV. THEOR.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, that is, divides them into two equal parts.

N. B. A parallelogram is a four sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.

Let ACDB be a $\square$, BC its diam^r: the opp. sides and $\angle^s$ of the fig. shall be = one another; and the diam^r BC shall bis^t it.

For,



∵ BC meets the ||s AB, CD,

29. 1. ∴ ∠ ABC = alt. ∠ BCD:

and ∵ BC meets the ||s AC, BD,

29. 1. ∴ ∠ ACB = alt. ∠ CBD:

Hence, in the two △s ABC, BCD,

∠s ABC, ACB = ∠s BCD, CBD, each to each,

and the adjt side BC is com. to both △s:

26. 1. ∴ $\begin{cases} \text{the third } \angle \text{ BAC} = \text{the third BDC,} \\ \text{side AB} = \text{CD, side AC} = \text{BD.} \end{cases}$

And,

∵ ∠ ABC = BCD, and ∠ CBD = ACB,

Ax. 2. ∴ the whole ∠ ABD = the whole ACD:

and, from above, ∠ BAC = BDC:

∴ *the opposite sides and angles of parallelograms are equal to one another.*

Also, in the two △s ABC, BCD,

side AB = CD, BC is com. to both,

and ∠ ABC = ∠ BCD;

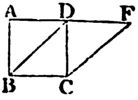
4. 1. ∴ △ ABC = △ BCD.

∴ *the parallelogram is bisected by its diameter* BC. [Q. E. D.]

PROP. XXXV. THEOR.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the ▱s ABCD, EBCF be on the same base BC, and between the same ||s AF, BC: ▱ ABCD = ▱ EBCF.

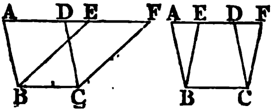


First, let the sides AD, DF, opp. to the base BC of the ▱s be each terminated in the same pt. D;

then each ▱ is double of the $\triangle$ BDC; 34. 1.
and $\therefore$ ▱ ABCD = ▱ DBCF. Ax. 6

But if the sides AD, EF be not terminated in the same p[t] D, then,

$\because$ ABCD is a ▱, 31. 1.
$\therefore$ AD = BC;
and
$\because$ EBCF is a ▱,
$\therefore$ EF = BC;
$\therefore$ AD = EF, Ax. 1
and DE = DE;



$\therefore \left\{\begin{matrix}\text{the whole, or the}\\ \text{rem}^{r}\text{ AE}\end{matrix}\right\} = \left\{\begin{matrix}\text{the whole, or the}\\ \text{rem}^{r}\text{ DF.}\end{matrix}\right.$ Ax. 2. or 3.

Hence, in the $\triangle$[s] EAB, FDC,
side AE = DF,
side AB = DC; 34. 1.
and ext[r] $\angle$ FDC = int[r] $\angle$ EAB: 29. 1.
$\therefore$ base EB = FC, and $\triangle$ EAB = $\triangle$ FDC. 4. 1.

From the trapezium ABCF, take the $\triangle$ FDC, and from the same fig. take the $\triangle$ EAB;
the rem[rs] will be = one another, Ax. 3.
i. e. ▱ ABCD = ▱ EBCF.

$\therefore$ *parallelograms on the same base, &c.*

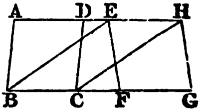
[Q. E. D.]

PROP. XXXVI. THEOR.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be ▱[s] on = bases BC, FG, and between the same $\|$[s] AH, BG

▱ ABCD = ▱ EFGH.



Join BE, CH; then,

$\because$ BC = FG,

Hyp. and FG = EH,

34. 1. $\therefore$ BC = EH;

Ax. 1. and these $|^{s}$ are $\|$, and are joined towards the same parts by the $|^{s}$ BE, CH:

but $|^{s}$, w^{h} join the extr^{s} of $|^{s}$ that are = and $\|$,

33. 1. towards the same parts, are themselves = and $\|$;

$\therefore$ EB, HC are = and $\|$,

Def. 34. 1. $\therefore$ EBCH is a ▱;

and it is on the same base BC, and between the same $\|^{s}$ BC, AH as is ▱ ABCD;

35. 1. $\therefore$ ▱ EBCH = ▱ ABCD:

for the like reasons,

▱ EBCH = EFGH;

Ax. 1. $\therefore$ ▱ ABCD = EFGH.

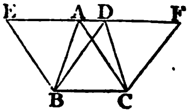
$\therefore$ *parallelograms, &c.* [Q. E. D.]

PROP. XXXVII. THEOR.

Triangles upon the same base, and between the same parallels, are equal to one another.

Let the △^{s} ABC, DBC be on the same base BC, and between the same $\|^{s}$ AD, BC:

△ ABC = △ DBC.



Post. 2. Prod. AD both ways;

31. 1. through B draw BE $\|$ CA;

through C draw CF $\|$ BD:

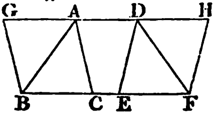
Def. 34. 1. then each of the fig^{s}. EBCA, DBCF is a ▱:

and these ▱^{s} are on the same base BC, and
between the same ||^s BC, EF;
∴ ▱ EBCA = ▱ DBCF:
But ∵ every ▱ is bis^d by its diam^r; 35 1.
∴ $\triangle$ ABC = $\frac{1}{2}$ ▱ EBCA, 34. 1.
$\triangle$ DBC = $\frac{1}{2}$ ▱ DBCF:
and the halves of = things are themselves =; Ax. 7.
∴ $\triangle$ ABC = $\triangle$ DBC.
∴ *triangles, &c.* [Q. E. D.]

PROP. XXXVIII. THEOR.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the $\triangle^s$ ABC, DEF be on = bases BC EF, and between the the same ||^s BF, AD:
$\triangle$ ABC = $\triangle$ DEF.



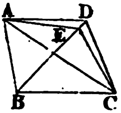
Prod. AD both ways; Post 2.
through B draw BG || CA; 31. 1.
through F draw FH || ED:

Then,
each of the fig^s GBCA, DEFH is a ▱; Def 34. 1.
and these ▱^{s} are on = bases BC, EF,
and between the same ||^s BF, GH;
∴ ▱ GBCA = ▱ DEFH: 36. 1.
But every ▱ is bis^d by its diam^r; 34. 1.
and ∴ $\triangle$ ABC = $\frac{1}{2}$ ▱ GBCA,
$\triangle$ DEF = $\frac{1}{2}$ ▱ DEFH;
and the halves of = things are themselves =; Ax. 7.
∴ $\triangle$ ABC = $\triangle$ DEF.
∴ *triangles, &c.* [Q. E. D.]

PROP. XXXIX. THEOR.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Let △ ABC = △ DBC, and let these △s be on the same base BC, and on the same side of it: they shall be between the same ||s.



Join AD; AD shall be || BC.

For, if not, through A draw

31. 1. AE || BC, and join EC.

Then,

the △s ABC, EBC are on the same base BC and between the same ||s BC, AE;

37. 1. and ∴, △ ABC = △ EBC:

Hyp. but △ ABC = △ DBC;

Ax. 1. ∴ △ DBC = △ EBC,

or the greater = the less,

wh is absurd;

∴ AE ∦ BC.

Similarly it may be shown that no other | but AD is || BC;

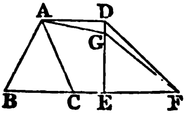
∴ AD is || BC

∴ *equal triangles, &c.* [Q. E. D.]

PROP. XL. THEOR.

Equal triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.

Let △ ABC = △ DEF, and let these △s be on = bases BC, EF, in the same |, and towards the same parts: they shall be between the same ||s



Join AD: AD shall be ∥ BC.

For, if not, through A draw AG ∥ BF, and join GF. 31. 1.

Then,

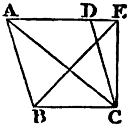
the $\triangle^s$ ABC, GEF are on = bases BC, EF,
and between the same $\|^s$ BF, AG;
and ∴ $\triangle$ ABC = $\triangle$ GEF; 38. 1
but $\triangle$ ABC = $\triangle$ DEF; Hyp.
∴ $\triangle$ GEF = $\triangle$ DEF, Ax. 1.
or the greater = the less.
w[h] is absurd;
∴ AG is $\nparallel$ BF:

And similarly it may be shown that no other ∥
but AD is ∥ BE;
∴ AD is ∥ BF.

∴ *equal triangles, &c.* [Q. E. D.]

PROP. XLI. THEOR.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.



Let ▱ ABCD and $\triangle$ EBC be upon the same base BC, and between the same $\|^s$ BC, AE: the ▱ shall be double of the $\triangle$.

Join AC; then,

∵ the $\triangle^s$ ABC, EBC are on the same base BC,
and between the same $\|^s$ BC, AE;

37. 1. $\therefore \triangle ABC = \triangle EBC$:

34. 1. But, every ▱ is bis[d] by its diam[r],

and $\therefore$ ▱ ABCD is double of $\triangle$ ABC:

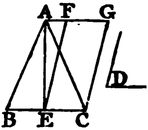
$\therefore$ ABCD is also double of $\triangle$ EBC.

$\therefore$ *if a parallelogram, &c.* [Q. E. D.]

PROP. XLII. PROB.

To describe a parallelogram that shall be equal a given triangle, and have one of its angles equ to a given rectilineal angle.

Let ABC be the given $\triangle$, and D the given $\angle$ it is req[d] to desc. a ▱ that shall be $=$ $\triangle$ AB(and have an $\angle = \angle$ D:



10. 1. Bis[t] BC in p[t] E, join AE, and at the p[t] E in the | EC

23. 1. make the $\angle$ CEF $= \angle$ D;

31. 1. through A draw AFG $\parallel$ EC,

Def. 34. 1. through C draw CG $\parallel$ EF: then, FECG is a ▱

Constr. And $\because$ BE $=$ EC, and BC is $\parallel$ AG,

38. 1. $\therefore \triangle ABE = \triangle AEC$;

and $\therefore \triangle$ ABC is double of $\triangle$ AEC:

But ▱ FECG and $\triangle$ AEC are on the sam base EC, and between the same $\parallel$[s] EC, AG

41. 1. and $\therefore$ ▱ FECG is double of $\triangle$ AEC:

Ax. 6. $\therefore$ ▱ FECG $= \triangle$ ABC,

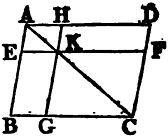
and its $\angle$ CEF $=$ the given $\angle$ D.

$\therefore$ *is described a parallelogram* FECG *equal* *the given triangle* ABC, *and having one of i angles equal to the given angle* D.

[Q. E. F.]

PROP. XLIII. THEOR.

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.



Let ABCD be a ▱, of wh AC is the diamr, EH, GF, ▱s about AC, i. e. through wh AC passes; BK, KD the other ▱s that make up the whole fig. ABCD, and wh are therefore called the *complements.*

The complt BK shall be = the complt KD.

For,

∵ ABCD is a ▱, and AC its diamr,

∴ △ABC = △ADC. 34. 1.

Again,

∵ AEKH is a ▱, and AK its diamr,

∴ △AEK = △AHK; 34. 1.

and, for the same reason,

△KGC = △KFC.

Hence, ∵ △AEK = △AHK,

and △KGC = △KFC;

∴ △s(AEK + KGC) = △s(AHK + KFC): Ax. 2.

But it was proved, that

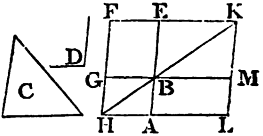
the whole △ABC = the whole △DAC;

∴ the remg complt BK = the remg complt KD. Ax. 3.

∴ *the complements, &c.* [Q. E. D.]

PROP. XLIV. PROB.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.



Let AB be the given $|$, C the given $\triangle$, and D the given $\angle$: it is req^{d} to apply to AB a ▱ that shall be $=$ C, and have an $\angle = $ D.

42. 1. Make the ▱ BEFG $= \triangle$ C, and having the $\angle$ EBG $= \angle$ D, and placed so that BE be in the same $|$ with AB; prod. FG to H : through A
31. 1. draw AH $\parallel$ BG or EF, and join HB.

Then,

$\because$ HF falls on the $\parallel^{s}$ AH, EF,

29. 1. $\therefore \angle^{s}(\text{AHF}+\text{HFE}) = $ two r^{t} $\angle^{s}$;

and $\therefore \angle^{s}(\text{BHF}+\text{HFE}) <$ two r^{t} $\angle^{s}$:

but $|^{s}$ w^{h}, with another $|$, make the int^{r} $\angle^{s}$ on the same side together $<$ two r^{t} $\angle^{s}$, do meet, if prod^{d}
Ax. 12. far enough :

$\therefore$ HB, FE, being prod^{d}, shall meet :

let them meet in K, through K draw KL $\parallel$ EA or FH, and prod. HA, GB to the p^{ts} L, M.

Then, FHLK is a ▱, of which the diam^{r} is HK, and AG, ME are ▱^{s} about HK; and LB, BF are the compl^{ts};

43. 1. and $\therefore$ LB $=$ BF :

Constr. but BF $= \triangle$ C,

Ax. 1. $\therefore$ LB $= \triangle$ C :

and ∵ ∠ GBE = ∠ ABM, 15. 1.
and also ∠ GBE = ∠ D, Constr.
∴ ∠ ABM = ∠ D. Ax. 1.

∴ *to the straight line* AB *is applied the parallelogram* LB *equal to the triangle* C, *and having the angle* ABM *equal to the angle* D.

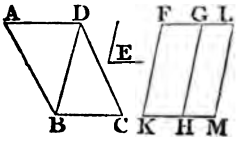
[Q. E. F.]

PROP. XLV. PROB.

To describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given rectilineal angle.

Let ABCD be the given fig. and E the given ∠ : it is req^d to desc. a ▱ that shall be = ABCD, and have an ∠ = E.

Join DB; desc. ▱ FH = △ ADB, and having the ∠ FKH = ∠ E; 42. 1.
and to | GH apply the ▱ GM = △ DBC, and having the ∠ GHM = ∠ E: the fig. FKML shall be the ▱ req^d. 44. 1.



For,
∵ ∠ E = each of the ∠^s FKH, GHM, Constr.
∴ ∠ FKH = ∠ GHM: Ax. 1.
let ∠ KHG be added to each;
then,
∠^s (FKH + KHG) = ∠^s (GHM + KHG): Ax. 2.
but ∠^s (FKH + KHG) = two r^t ∠^s; 29. 1.
∴ ∠^s (GHM + KHG) = two r^t ∠^s: Ax. 1.

Thus, at the p[t] H in | HG, the two |[s] KH, HM on the opp. sides of it, make the adj[t] $\angle$[s] = two r[t] $\angle$

14. 1. and $\therefore$ KH is in the same | with HM.

Again,

$\because$ | HG meets the ||[s] KM, FG,

29. 1. $\therefore$ $\angle$ MHG = the alt. $\angle$ HGF:

let $\angle$ HGL be added to each;

then,

Ax. 2. $\angle$[s](MHG + HGL) = $\angle$[s](HGF + HGL):

29. 1. but $\angle$[s](MHG + HGL) = two r[t] $\angle$[s];

Ax. 1. $\therefore$ $\angle$[s](HGF + HGL) = two r[t] $\angle$[s];

14. 1. and $\therefore$ FG is in the same | with GL:

Constr. And $\because$ KF is || HG, and HG || ML,

30. 1. $\therefore$ KF is || ML:

Constr. also KM is || FL;

Def. 34. 1. $\therefore$ KFLM is a ▱.

Constr. And $\because$ △ ABD = ▱ HF,

and △ DBC = ▱ GM,

Ax. 2. $\therefore$ the whole fig. ABCD = the whole ▱ KFLM

$\therefore$ *is described the parallelogram* KFLM *equal to the given rectilineal figure* ABCD, *and having the angle* FKM *equal to the angle* E.

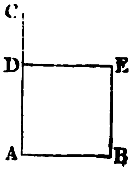
[Q. E. F.]

Cor.—From this it is manifest how, to a given | to apply a ▱ w[h] shall be = a given rect[l] fig. and have an $\angle$ = a given $\angle$: viz. by applying to the given | a ▱ that is = the first triangle ABD and has an $\angle$ = the given $\angle$.

PROP. XLVI. PROB.

scribe a square upon a given straight line.

AB be the given |; it is
desc. a sq. on AB.



n A draw AC at r^t ∠^{s} 11. 1.
and make AD = AB; 3. 1.
ugh D draw DE ∥ AB, 31. 1.
ugh B draw BE ∥ AD:
ADEB is a ▱; Def. 34. 1.
and AB = DE, 34. 1.
AD = BE;
but AB = AD; Constr
∴ AB = AD = DE = EB, Ax. 1.
and the ▱ ADEB is equilat^l.

AD meets the ∥^{s} AB, DE,
∠^{s}(BAD + ADE) = two r^t ∠^{s}; 29. 1.
but BAD is a r^t ∠; Constr.
∴ ADE is also a r^t ∠: Ax. 3.
and the opp. ∠^{s} of ▱^{s} are =; 34. 1.
each of the ∠^{s} ABE, BED is a r^t ∠;
the fig. ADEB is rectangular,
has been shown to be equilat^l.

is a square, and it is described on AB. Def. 30.

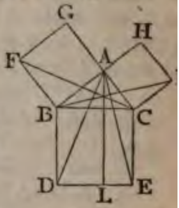
[Q. E. F.]

—Hence, if a ▱ have one r^t ∠, all its
r^t ∠^{s}.

PROP. XLVII. THEOR.

In any right-angled triangle, the square which described upon the side subtending the right ang is equal to the squares described upon the sic which contain the right angle.

Let ABC be a r' $\angle^d \triangle$, BAC being the r^t $\angle$ the sq. desc^d on the side BC shall be = the sq^s desc^d on the sides AB, AC.



64. I. On BC desc. the sq. BDEC; on BA, AC, the sq^s GB, HC; through A

31. I. draw AL ∥ BD or CE, and join AD, FC.

Hyp. Then, ∵ $\angle$ BAC is a r^t $\angle$,

Def. 30. and also, $\angle$ BAG is a r^t $\angle$,

∴ the two |^s AC, AG on the opp. sides of A

14. I. make with it, at A, the adj^t $\angle^s$ = two r^t $\angle^s$;

and ∴ CA is in the same | with AG:

for the same reason,

AB is in the same | with AH.

Again,

Def. 30. ∵ each of the $\angle^s$ DBC, FBA is a r^t $\angle$,

Ax. 1. ∴ $\angle$ DBC = $\angle$ FBA:

let $\angle$ ABC be added to each;

Ax. 2. then, the whole $\angle$ DBA = the whole $\angle$ FB(

hence, in $\triangle^s$ ABD, FBC,

Def. 30. ∵ { side AB = FB, BD = BC, and $\angle$ DBA = $\angle$ FBC;

4. I. ∴ { the base AD = the base FC, and $\triangle$ ABD = $\triangle$ FBC

Now, the ▱ BL and the △ ABD are on the same base BD, and between the same $\parallel^s$ BD, AL,

and ∴ ▱ BL is double of △ ABD: 41. 1.

Also, the sq. GB and the △ FBC are on the same base FB, and between the same $\parallel^s$ FB, GC;

and ∴ sq. GB is double of △ FBC:

but, from above

△ ABD = △ FBC;

and the doubles of = things are themselves =; Ax. 6.

∴ ▱ BL = the sq. GB.

In the same manner, by joining AE, BK, it can be shown that

▱ CL = sq. HC;

∴ the whole sq. BDEC = the two sq^s GB, HC; Ax. 2.

and the sq. BDEC is desc^d on BC,

the sq^s GB, HC on BA, AC,

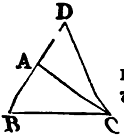
∴ the sq. on BC = the sq^s on BA, AC.

∴ *in any right-angled triangle, &c.* [Q. E. D.]

PROP. XLVIII. THEOR.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the sq. desc^d on BC, a side of △ ABC, be = the sq^s desc^d on the other two sides AB, AC: ∠ BAC shall be a r^t ∠.



From A draw AD at r^t ∠^{s} to AC, making AD = AB, and join DC. 11. 1 3. 1.

Then,

$\because$ DA = AB,

$\therefore$ $DA^2 = AB^2$:

let AC^2 be added;

Ax. 2. then, $DA^2 + AC^2 = AB^2 + AC^2$:

but $\because$ DAC is a $r^t \angle$,

47. 1. $\therefore$ $DC^2 = AD^2 + AC^2$:

Hyp. also, $BC^2 = AB^2 + AC^2$;

Ax. 1. $\therefore$ $DC^2 = BC^2$;

and $\therefore$ DC = BC.

Thus, in $\triangle^s$DAC, BAC,

$\because$ $\begin{cases} \text{side AD = AB, AC is com. to both} \\ \text{and base DC = base BC,} \end{cases}$

8. 1. $\therefore$ $\angle$ DAC = $\angle$ BAC:

Constr. but DAC is a $r^t \angle$;

Ax. 1. and $\therefore$ BAC is also a $r^t \angle$.

$\therefore$ *if the square, &c.* [Q. E. D.

BOOK II.

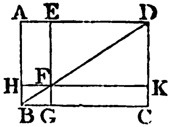
DEFINITIONS.

I.

Every right-angled parallelogram, *or rectangle*, is said to be contained by any two of the straight lines which contain one of the right angles.

II.

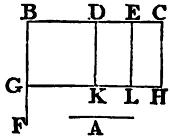
In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. 'Thus the parallelogram HG, together with 'the complements AF, FC, 'is the gnomon, which is 'more briefly expressed by the letters AGK, or 'EHC, which are at the opposite angles of the 'parallelograms *which* make the gnomon.'



PROP. I. THEOR.

If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Of the two $|^s$ A and BC, let one BC be div^d into any n^o of parts in the p^{ts} D, E: the rect. contained by the two $|^s$ A, BC shall be $=$ the rect. contained by A, BD, $+$ that contained by A, DE, $+$ that contained by A, EC:



or $\quad$ A. BC $=$ A. BD $+$ A. DE $+$ A. EC.

11. 1. From p^t B draw BF at r^t $\angle^s$ to BC, and make
3. 1. BG $=$ A; through G draw GH $\parallel$ BC; and through
31. 1. D, E, C, draw DK, EL, CH $\parallel$ BG. Then, the rect. BH $=$ the rect^s (BK $+$ DL $+$ EH).

But BH is contained by the $|^s$ GB, BC,
Constr. of w^h GB $=$ A,
and ∴ BH $=$ A. BC:

Also,
BK is contained by GB, BD, of w^h GB $=$ A,
and ∴ BK $=$ A. BD:
34. 1. DL is contained by DK, DE, of w^h DK $=$ BG $=$ A,
and ∴ DL $=$ A. DE:
similarly, EH $=$ A. EC:
∴ the rect. A. BC $=$ A. BD $+$ A. DE $+$ A. EC

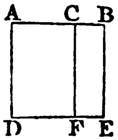
∴ *if there be two straight* lines, &c. [Q. E. D.]

PROP. II. THEOR.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.

Let | AB be div^d into any two parts in p^t C: the rect. AB, AC + the rect. AB, CB = the sq. of AB;

Or $AB.\ AC + AB.\ CB = AB^2$.



On AB desc. the sq. ADEB, and 46. 1.
through C draw CF ∥ AD or BE. 31. 1

Then,

AE = the rect^s (AF + CE):

but, $AE = AB^2$:

Also,

AF is contained by |^s AD, AC, of w^h AD = AB, Def. 30.
and ∴ AF = AB. AC;

CE is contained by |^s BE, CB, of w^h BE = AB,
and ∴ CE = AB. CB.

∴ the rect^s $(AB.\ AC + AB.\ CB) = AB^2$.

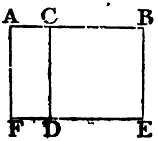
∴ *if a straight line, &c.* [Q. E. D.]

PROP. III. THEOR.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let | AB be div^d into any two parts in p^t C:
then, $AB.\ BC = AC.\ CB + BC^2$.

46. 1. On BC desc. the sq. CDEB; prod. ED to F, and through A
31. 1 draw AF || CD or BE.



Then,

$AE = AD + DB$:

But,

Def. 30. AE is contained by AB, BE, of w[h] $BE = BC$,

and $\therefore$ $AE = AB.BC$:

Also,

AD is contained by AC, CD of w[h] $CD = BC$.

and $\therefore$ $AD = AC.BC$:

Hyp. and $DB = BC^2$:

$\therefore$ $AB.BC = AC.BC + BC^2$.

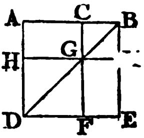
$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. IV. THEOR.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let | AB be div[d] into any two parts in p[t] C: then, $AB^2 = AC^2 + CB^2 + 2AC.CB$.

46. 1. On AB desc. the sq. ADEB; join BD; through C draw CGF ||
31. 1. AD or BE, and through G draw HK || AB or DE.



Then,

$\because$ BD falls on the ||[s] CF, AD,

29. 1. $\therefore$ ext[r] $\angle$ BGC = int[r] $\angle$ ADB ·

But $\because$ ADEB is a square,
$\therefore$ side $AB =$ side AD; Def. 30.
and $\therefore$ $\angle ADB = \angle ABD$: 5. 1.
$\therefore$ $\angle BGC = \angle CBG$; Ax. 1.
and $\therefore$ side $BC =$ side CG: 6. 1.

but also, $BC = GK$, $CG = BK$; 34. 1.
$\therefore BC = CG = GK = BK$,
and $\therefore$ the fig. CGKB is equilat[l]. Ax. 1.

Again,
$\because$ CB meets the $\|^s$ CG, BK,
$\therefore \angle^s (KBC + GCB) =$ two $r^t \angle^s$: 29. 1.
but KBC is itself a $r^t \angle$; Def. 30.
$\therefore$ GCB is also a $r^t \angle$; Ax. 3.
$\therefore$ the opp. $\angle^s$ CGK, GKB, are also $r^t \angle^s$; 34. 1.
$\therefore$ the fig. CGKB is rectangular:
and it has been shown to be equilat[l].
$\therefore$ it is a square;
and it is on the side CB. Def. 30.

For the same reasons,
HF is a square;
and it is on the side HG,
and $HG = AC$; 34. 1.
$\therefore$ HF, CK are the sq[s] of AC, CB.
And, $\because$ compl[t] $AG =$ compl[t] GE; 43. 1.
and that compl[t] $AG = AC.\ CG$
$= AC.\ CB$, Def. 30
$\therefore$ $GE = AC.\ CB$: Ax. 1.
and $\therefore$ $AG + GE = 2AC.\ CB$;
and HF, CK are the sq[s] of AC. CB:

$$\therefore \left\{ \begin{matrix} \text{the fig}^s\ (HF + CK + \\ AG + GE) \end{matrix} \right\} = \left\{ \begin{matrix} AC^2 + BC^2 \\ + 2AC.\ CB: \end{matrix} \right.$$

but HF, CK, AG and GE make up the
Ax. 1. fig. ADEB; and this fig. is the sq. of AB:

$\therefore AB^2 = AC^2 + BC^2 + 2AC.CB.$

$\therefore$ *if a straight line, &c.* [Q. E.

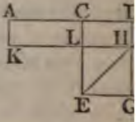
COR.—It is manifest, from the demonstr that ▱s about the diamr of a sq. are thems sqs.

PROP. V. THEOR.

If a straight line be divided into two equal and also into two unequal parts; the rect contained by the unequal parts, together the square of the line between the poin section, is equal to the square of half the li

Let | AB be divd into two = parts in the and into two ≠ parts in the pt D: then,

$AD.DB + CD^2 = CB^2.$



46. 1. On CB. desc. the sq. CEFB; join BE; through D draw
31. 1. DHG ∥ CE or BF; through H draw KLM ∥ CB or EF, and through A draw AK ∥ CL or BM. Then,

43. 1. the complt CH = the complt HF;
let DM be added to each;
Ax. 2. then, the whole CM = the whole DF.
Hyp. But, $\because AC = CB,$
36. 1. $\therefore AL = CM;$
Ax. 1. $\therefore AL = DF;$
let CH be added;

then, the whole $AH = (DF + CH)$, Ax. 2.
But, AH is contained by AD, DH, Def. 30.
of wh $DH = DB$, Cor.4.2.
and ∴ $AH = AD.\ DB$.
Also,
DF and CH make up the gnomon CMG;
∴ the gnomon CMG = the rect. AD. DB: Ax. 1.
add LG, *i. e.* CD^2; Cor.4.2.
then, $CMG + LG = AD.\ DB + CD^2$: 34. 1.
But CMG and LG together make up the fig. Ax. 2.
CEFB, wh is the sq. of CB;
∴ $AD.\ DB + CD^2 = CB^2$.
∴ *if a straight line, &c.* [Q.E.D.]

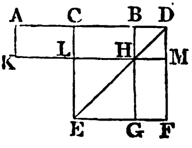
From this proposition it is manifest, that the difference of the sqs of two ≠ |s AC, CD, is = to the rect. contained by their sum and difference.

PROP. VI. THEOR.

If a straight line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let | AB be bisd in C, and prodd to D: then,

$$AD.\ DB + CB^2 = CD^2.$$



On CD desc. the sq. CEFD. 46. 1.
join DE; through B draw BHG ∥ CE or DF; through H draw KLM ∥ AD or EF; 31. 1.
and through A draw AK ∥ CL or DM.

Hyp. Then, $\because$ $AC = CB$,

36. 1. $\therefore$ rect. $AL =$ rect. CH;

43. 1. but $CH = HF$;

Ax. 1. $\therefore$ also, $AL = HF$:

let CM be added; then,

Ax. 2. the whole $AM =$ the gnomon CMG:

Def. 30. but $AM = AD.\,DM$

Cor. 4. 2. $= AD.\,DB$;

Ax. 1. $\therefore$ $CMG = AD.\,DB$:

Cor. 4. 2. let LG *i. e.* CB^2, be added,

Def. 30. Ax. 1. 2. then $AD.\,DB + CB^2 = CMG + LG$:

but CMG and LG make up the whole fig. $CEFD$

Constr. 1. and this fig. is CD^2;

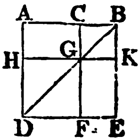
Ax. 2. $\therefore$ $AD.\,DB + CB^2 = CD^2$.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. VII. THEOR.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let | AB be div^d^ into any two parts in the p^t^ C: then, $AB^2 + BC^2 = 2\,AB.\,BC + AC^2$.



46. 1. On AB desc. the sq. $ADEB$, and make the same construction as in the preceding propositions.

Then,

43. 1. compl^t^ $AG =$ compl^t^ GE:

add CK; then,

$AK = CE$,

and $\therefore$ $AK + CE = 2\,AK$.

But AK, CE make up the gnomon AKF, together with the sq. CK;

and $\therefore$ AKF + CK = 2 AK: Ax. 1.

But, 2 AK = 2 AB. BK Def. 30.

= 2 AB. BC: Cor.4.3.

$\therefore$ AKF + CK = 2 AB. BC: Ax. 1.

add HF, *i. e.* AC^2; Cor.4.2. & 34.1.

then,

$AKF + CK + HF = 2\, AB.BC + AC^2$: Ax. 2.

but the gnomon AKF, together with the sq^s CK, HF, make up the whole fig. ADEB and that CK, and these fig^s. are the sq^s of AB, BC;

$\therefore\ AB^2 + BC^2 = 2\, AB.\, BC + AC^2$.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

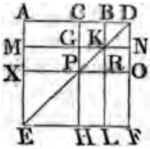
PROP. VIII. THEOR.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square of the other part, is equal to the square of the straight line, which is made up of the whole and that part.

Let | AB be div^d into any two parts in the p^t C:

$4\, AB.\, BC + AC^2 = \left\{ \begin{array}{l} \text{the sq. of the | made up of} \\ \text{AB and BC together.} \end{array} \right.$

Prod. AB to D, so that BD = BC; on AD, desc. the sq. AEFD; and construct two fig^s such as in the preceding propositions. Post. 2. 3. 1. 46. 1.



Constr. Then, $\because$ CB = BD,

34. 1 and that CB = GK, BD = KN,

Ax. 1. $\therefore$ GK = KN :

similarly, PR = RO :

And,

$\because$ CB = BD, GK = KN

36. 1. $\therefore$ rect. CK = BN, GR = RN ;

But CK, RN, are the compl[ts] of ▱ CO

43. 1. and $\therefore$ CK = RN ;

Ax. 1. $\therefore$ also, BN = GR ;

$\therefore$ BN = CK = GR = RN ;

and $\therefore$ the sum of these four = rect[s] is quadruple of one of them CK ;

Again,

Constr. $\because$ CB = BD

34. 1. and that CB = GK

Cor. 4. 2. Def. 30. = GP

and also, BD = BK

31. 1. = CG

$\therefore$ CG = GP :

and $\because$ CG = GP, PR = RO,

36. 1. $\therefore$ rect. AG = MP, PL = RF :

but MP, PL are the compl[ts] of ▱ ML,

43. 1. $\therefore$ MP = PL ;

Ax. 1. $\therefore$ also, AG = RF :

$\therefore$ AG = MP = PL = RF :

and $\therefore$ the sum of these four = rect[s] is quadruple of one of them AG.

And from above,

the sum of BN, CK, GR, RN is quadruple of CK

$\therefore$ the eight rect[s] w[h] form the gnomon AOH are together quadruple of AK.

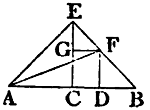
and $\because$ rect. $AK = AB.BK$,
$= AB.BC$,
$\therefore$ $4AK = 4AB.BC$.
but, from above, $4AK = AOH$.
$\therefore$ $4AB.BC = AOH$:
add XH, *i. e.* AC^2; then, Ax. 1.
$4AB\,BC + AC^2 = AOH + XH$: Cor.4.2 & 34.1. Ax. 2.
but AOH and XH make up the fig. AEFD.
and this fig. is AD^2:
$\therefore 4AB.BC + AC^2 = AD^2$.
$= (AB + BC)^2$. Constr

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. IX. THEOR.

If a straight line be divided into two equal, and also into two unequal parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let | AB be div[d] into two = parts at the p[t] C and into two $\neq$ parts at D: then
$AD^2 + DB^2 = 2(AC^2 + CD^2)$.



From C draw CE at r[t] $\angle$[s] to AB and = AC or CB; join EA, EB; through D draw DF $\parallel$ CE; through F draw FG $\parallel$ BA, and join AF. Then, 11.1. 3.1. 31.1.
$\because$ $AC = CE$, Constr.
$\therefore$ $\angle EAC = \angle AEC$: 5.1.

and $\because$ ACE is a $r^t \angle$,

32. 1. $\therefore \angle^s(AEC+EAC)=$ one $r^t \angle$;

and these $\angle^s$ are $=$ one another;

$\therefore$ each of them is half a $r^t \angle$:

similarly,

each of the $\angle^s$ CEB, EBC is half a $r^t \angle$

$\therefore$ the whole AEB is a $r^t \angle$.

And, $\because$ GEF is half a $r^t \angle$,

29. 1. and $EGF=$ $int^r \angle$ $ECB=$ a $r^t \angle$,

32. 1. $\therefore$ $rem^g \angle$ EFG is half a $r^t \angle$:

$\therefore$ $\angle GEF = \angle EFG$,

6. 1. and $\therefore$ side $EG=$ side GF.

Again,

$\because$ $\angle$ FBD is half a $r^t \angle$,

29. 1. and $\angle FDB=$ $int^r \angle$ $ECB=$ a $r^t \angle$;

$\therefore$ $rem^g \angle$ BFD is half a $r^t \angle$:

$\therefore$ $\angle FBD = \angle BFD$,

6. 1. and $\therefore$ side $DF=$ side DB.

And,

$\because$ $AC=CE$,

$\therefore$ $AC^2=CE^2$,

and $\therefore$ $AC^2+CE^2=2\,AC^2$:

but $\because$ ACE is a $r^t \angle$,

47. 1. $\therefore$ $AE^2=AC^2+CE^2$;

$\therefore$ $AE^2=2\,AC^2$.

Again,

$\because$ $EG=GF$,

$\therefore$ $EG^2=GF^2$;

and $\therefore$ $EG^2+GF^2=2\,GF^2$:

47. 1. but $EF^2=EG^2+GF^2$;

$\therefore$ $EF^2=2\,GF^2$;

34. 1. $=2\,CD^2$:

and, from above, $AE^2 = 2\,AC^2$;

$\therefore\ AE^2 + EF^2 = 2\,(AC^2 + CD^2)$.

But $\because$ AEF is a r' $\angle$,

$\therefore\ AF^2 = AE^2 + EF^2$; 47. 1.

and $\therefore\ AF^2 = 2\,(AC^2 + CD^2)$.

But $\because$ ADF is a r' $\angle$,

$\therefore\ AF^2 = AD^2 + DF^2$; 47. 1.

$\therefore\ AD^2 + DF^2 = 2\,(AC^2 + CD^2)$.

and $DF = DB$;

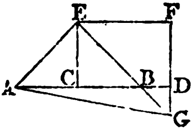
$\therefore\ AD^2 + DB^2 = 2\,(AC^2 + CD^2)$.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. X. THEOR.

If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let | AB be bis^d in the p^t C and prod^d to D; then, $AD^2 + DB^2 = 2\,(AC^2 + CD^2)$.



From C draw CE at r^t $\angle^s$ to AB and = AC or CB; join AE, EB; through E draw EF || AB, and through D draw DF || CE. 11. 1. 3. 1. 31. 1.

Then,

$\because$ | EF meets the ||^s EC, FD,

$\therefore\ \angle^s\,(CEF + EFD) =$ two r^t $\angle^s$; 29. 1

and $\therefore\ \angle^s\,(BEF + EFD) <$ two r^t $\angle^s$;

Ax. 12. but $|^s$ w^h, with another $|$, make the int^r $\angle^s$ on the same side together $<$ two r^t $\angle^s$, will meet if pro far enough :

$\therefore$ EB, FD will, if prod^d, meet towards B, I let them meet in G, and join AG.

Then,

$\because$ $AC = CE$,

5. 1. $\therefore$ $\angle AEC = \angle CAE$;

and ACE is a r^t $\angle$;

32. 1. $\therefore$ each of the $\angle^s$ AEC, CAE is half a r^t $\angle$

For the same reason,

each of the $\angle^s$ CEB, CBE is half a r^t $\angle$:

$\therefore$ AEB is a r^t $\angle$.

And,

$\because$ EBC is half a r^t $\angle$,

15. 1. $\therefore$ its opp. $\angle$ DBG is also half a r^t $\angle$;

29. 1. but $BDG =$ alt. $\angle DCE =$ a r^t $\angle$;

$\therefore$ rem^g $\angle$ DGB is half a r^t $\angle$,

$\therefore$ $\angle DGB = \angle DBG$,

6. 1. and $\therefore$ side $BD =$ side DG.

Again,

$\because$ EGF is half a r^t $\angle$,

34. 1. and that $\angle EFG =$ opp. $\angle ECD =$ a r^t $\angle$

$\therefore$ rem^g $\angle$ FEG is half a r^t $\angle$;

$\therefore$ $\angle FEG = \angle EGF$,

6. 1. and $\therefore$ side $FG =$ side FE.

And, $\because$ $EC = AC$,

$\therefore$ $EC^2 = AC^2$

and $\therefore$ $EC^2 + AC^2 = 2\,AC^2$:

47. 1. but $AE^2 = EC^2 + AC^2$;

$\therefore$ $AE^2 = 2\,AC^2$.

Again

$\because$ $FG = FE$,

$\therefore$ $FG^2 = FE^2$;

and $\therefore$ $FG^2 + FE^2 = 2\,FE^2$:

but $EG^2 = FG^2 + FE^2$; 47. 1.

$\therefore$ $EG^2 = 2\,FE^2$

$= 2\,CD^2$: 34. 1.

and, from above, $AE^2 = 2\,AC^2$;

$\therefore$ $AE^2 + EG^2 = 2\,(AC^2 + CD^2)$:

but $AG^2 = AE^2 + EG^2$; 47. 1.

$\therefore$ $AG^2 = 2\,(AC^2 + CD^2)$

but also, $AG^2 = AD^2 + GD^2$ 47. 1.

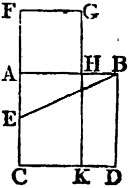
$= AD^2 + BD^2$;

$\therefore$ $AD^2 + BD^2 = 2\,(AC^2 + CD^2)$.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. XI. PROB.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.



Let AB be the given |: it is reqd to div. it into two parts, so that the rect. contained by the whole and one of the parts shall be = the sq. of the other part.

On AB desc. the sq. ABDC; 46. 1.
bist AC in E, and join BE; prod. 10. 1.
CA to F, making EF = EB, and 3. 1.
on AF desc. the sq. FGHA: 46. 1.

AB shall be div^d in H, so that $AB . BH = AH^2$

Prod. GH to K: then,

$\because$ AC is bis^d in E, and prod^d to F,

6. 2. $\therefore$ $CF . AF + AE^2 = EF^2$

Constr. $= EB^2$

47. 1. $= AB^2 + AE^2$;

take away the com. part AE^2;

Ax. 3. then, $CF . AF = AB^2$;

But, fig. $FK = CF . FG$

Def. 30. $= CF . AF$,

fig. $AD = AB^2$;

Ax. 1. and $\therefore$ fig. $AD = FK$:

take away the com. part AK;

Ax. 3. then, the rem^r FH = the rem^r HD:

but $HD = HB . BD$

Def. 30. $= HB . AB$;

$FH = AH^2$;

and $\therefore$ $AB . HB = AH^2$.

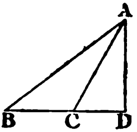
$\therefore$ *the straight line* AB *is divided in* H *so that the rectangle* AB. BH *is equal to the square of* AH.

[Q. E. F.]

PROP. XII. THEOR.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and

the straight line intercepted without the triangle between the perpendicular and the obtuse angle.



Let ABC be an obt. $\angle^d \triangle$;
and, ACB being the obt. $\angle$,
from A draw AD $\perp$ BC prod^d: 12. 1.
AB^2 shall be $> (AC^2 + BC^2)$
by twice the rect. BC. CD.

For,

$\because$ | BD is div^d into two parts in C

$\therefore BD^2 = BC^2 + CD^2 + 2\ BC.CD$: 4. 2.

add AD^2; then,

$BD^2 + AD^2 = BC^2 + CD^2 + AD^2 + 2\ BC.CD$: Ax. 2.

but, $\because$ ADB is a r^t $\angle$,

$\therefore AB^2 = BD^2 + AD^2$, 47. 1.

and also,

$AC^2 = CD^2 + AD^2$; 47. 1.

$\therefore AB^2 = BC^2 + AC^2 + 2\ BC.CD$,

i.e. the sq. of AB exceeds the sq^s of AC, BC by twice the rect. BC. CD.

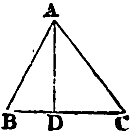
$\therefore$ *in obtuse-angled triangles, &c.* [Q. E. D.]

PROP. XIII. THEOR.

In every triangle, the square of the side subtending either of the acute angles is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

Let ABC be any $\triangle$, $\angle$ at B one of its acute $\angle^s$; and on BC, one of the sides containing this $\angle$,

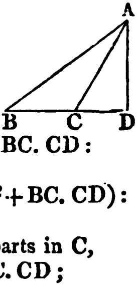
12.1. let fall the $\perp$ AD from the opp. $\angle$ BAC:
AC^2 shall be $< (AB^2+BC^2)$
by twice the rect. BC. BD.



First,
let AD fall within $\triangle$ ABC;

then,
$\because$ | CB is div[d] into two parts in p[t] D,
7.2. $\therefore CB^2+BD^2=CD^2+2\ BC.BD$:
let AD^2 be added; then,
Ax.2. $CB^2+BD^2+AD^2=CD^2+AD^2+2\ BC.BD$:
but, $\because$ each of the $\angle$s at D is a r[t] $\angle$,
47.1. $\therefore$ $AB^2=BD^2+AD^2$,
and $AC^2=CD^2+AD^2$;
$\therefore CB^2+AB^2=AC^2+2\ BC.BD$.
i.e. AC^2 is $<(BC^2+AB^2)$ by 2 BC. BD.

Secondly,
let AD fall without $\triangle$ ABC:
then,



$\because$ $\angle$ at D is a r[t] $\angle$,
16.1. $\therefore$ $\angle$ ACB is $>$ a r[t] $\angle$;
12.2. and $\therefore$ $AB^2=AC^2+BC^2+2\ BC.CD$:
let BC^2 be added;
Ax.2. then, $AB^2+BC^2=AC^2+2\,(BC^2+BC.CD)$:
But,
$\because$ BD is div[d] into two parts in C,
3.2. $\therefore$ $BD.BC=BC^2+BC.CD$;
and
the doubles of equal things are themselves equal;
$\therefore AB^2+BC^2=AC^2+2\ BD.BC$:
i.e. AC^2 is $<(AB^2+BC^2)$ by 2 BD. BC.

Lastly, let the side AC be $\perp$ to BC: then BC is the | between the $\perp$ and the acute $\angle$ at B: and,

$\because \quad AB^2 = AC^2 + BC^2,$ 47. 1.

$\therefore AB^2 + BC^2 = AC^2 + 2BC^2.$

$= AC^2 + 2BC.BC$

$\therefore$ *in every triangle, &c.* [Q. E. D.]

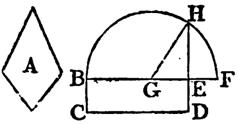
PROP. XIV. PROB.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rect[l] fig.: it is req[d] to desc. a sq. that shall be $=$ A.

Desc. the r[t] $\angle$ [d] $\square$ BCDE $=$ A: then, if its sides 45. 1. BE, ED, be equal, the fig. is a sq., and what was Def. 30. req[d] is done.

But if these sides be $\neq$, prod. one of them BE to F, make EF $=$ ED, and 3. 1. bis[t] BF in G; from cent. 10. 1. G, at dist. GB or GF, desc. the $\frac{1}{2}$ $\odot$ BHF, and prod. DE to H. The sq. desc[d]. on EH shall be $=$ the given fig. A.



Join GH: then,

$\because$ | BF is div[d] into two equal parts in p[t] G, and into two unequal parts in p[t] E,

5. 2. $\therefore$ BE. EF + $EG^2 = GF^2$

Def. 16. $= GH^2$

47. 1. $= EH^2 + EG^2$:

take away the com. part EG^2;

Ax. 3. then the rem^r BE. EF = the rem^r EH^2:

Constr. but the ▱ BD = BE. ED

Ax. 1. = BE. EF;

$\therefore$ BD = EH^2:

but BD = the fig. A,

$\therefore EH^2 = A$.

And $\therefore$ *there is found a square equal to the given figure* A, *viz. the square described on* EH.

[Q. E. I.]

BOOK III.

DEFINITIONS.

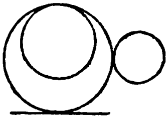
I

EQUAL circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

"This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal."

II.

A straight line is said to touch a circle when it meets the circle, and being produced does not cut it.

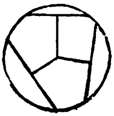


III.

Circles are said to touch one another, which meet but do not cut one another

IV.

Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



V.

And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

VI.

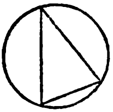
A segment of a circle is the figure contained by a straight line and the circumference which it cuts off.

VII.

"The angle of a segment is that which is contained by the straight line and the circumference."

VIII.

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.

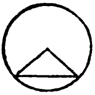


IX.

And an angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.

X.

A sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



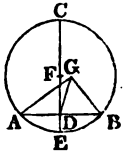
XI.

Similar segments of circles are those in which the angles are equal, or which contain equal angles.

PROP. I. PROB.

To find the centre of a given circle.

Let ABC be the given ⊙: it is req^d to find its cent.



Within the ⊙ draw any | AB and bis^t it in D; from the p^t D draw DC at r^t ∠^s to AB, prod. CD to E. and bis^t CE in F: 10. 1. 11. 1.

F shall be the cent. of ⊙ ABC

For, if not, let, if possible, some other p^t G be the cent.; and join GA, GD, GB:

Then, in △^s ADG, BDG,

∵ { side DA = DB, DG is com. to both, Constr.
and also rad. AG = rad. BG, Def. 15.

∴ ∠ ADG = BDG: 8. 1.

but when one |, standing upon another |, makes the adj^t ∠^s = one another, each ∠ is a r^t ∠; Def. 10.

∴ ∠ GDB is a r^t ∠:

but ∠ FDB is also a r^t ∠; Constr.

∴ ∠ FDB = ∠ GDB, Ax. 1.

i. e. the greater = the less,

w^h is impossible:

∴ G is not the centre of ⊙ ABC.

And in the same manner it can be shown that no other p^t but F is the cent. of the ⊙.

∴ *F is the centre.* [Q. E. I.]

Cor.—From this it is manifest, that if in a ⊙ one | bis^t another at r^t ∠^s, the cent. of the ⊙ is in that | w^h bis^ts the other.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let A, B be any two p^{ts} in the $\odot^{ce}$ of $\odot$ ABC: the | drawn from A to B shall fall within the $\odot$.

For, if not, let it, if possible, fall without the $\odot$, as AEB: find the
1. 3. cent. D of the $\odot$ ABC; join DA, DB; in the arc AB take any p^{t} F, join DF, and prod. DF to E.

Def. 15. Then, $\because$ rad. DA = rad. DB,
5. 1. $\therefore \angle$ DAB = $\angle$ DBA:
and $\because$ side AE of $\triangle$ ADE is prod^{d} to B,
16. 1. $\therefore$ ext^{r} $\angle$ DEB > int^{r} $\angle$ DAE:
but, from above,
$\angle$ DAE = $\angle$ DBE;
$\therefore \angle$ DEB > $\angle$ DBE;
19. 1. but to the greater $\angle$ the greater side is opp.
and $\therefore$ side DB > DE:
Def. 15. 1. but DB = DF;
$\therefore$ DF > DE,
i. e. the less > the greater,
w^{h} is impossible:
$\therefore$ | AB does not fall without the $\odot$.

In the same manner it may be shown that it does not fall upon the $\odot^{ce}$, and $\therefore$ it falls within it.

$\therefore$ *if any two points, &c.* [Q. E. D.]

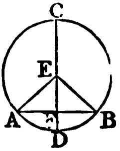
PROP. III. THEOR.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and if it cut it at right angles, it shall bisect it.

In $\odot$ ABC, let CD, a | drawn through the cent. bis^t any | AB, w^h does not pass through the cent. in p^t F: CD shall cut AB at r^t $\angle^s$.

Find E the cent. of the $\odot$, and join EA, EB. 1. 3.

Then, in $\triangle^s$ AFE, BFE,



$\because$ { AF = FB, Hyp.
FE is com.
and base AE = BE, } Def. 15. 1.

$\therefore$ $\angle$ AFE = $\angle$ BFE: 8. 1.

but when one |, standing on another |, makes the adj^t $\angle^s$ = one another,
each of them is a r^t $\angle$; Def. 10. 1.

$\therefore$ each of the $\angle^s$ AFE, BFE is a r^t $\angle$:

and $\therefore$ | CD drawn through the cent. and bisecting another | AB, w^h does not pass through the cent. cuts AB at r^t $\angle^s$.

But, let CD cut AB at r^t $\angle^s$: CD shall also bis^t AB, *i.e.* AF = BF. Make the same constr^n;

Then, $\because$ rad. EA = rad. EB, Def. 15. 1.
$\therefore$ $\angle$ EAF = EBF; 5. 1.
and r^t $\angle$ AFE = r^t $\angle$ BFE: Def. 10. 1.

Hence, in the two $\triangle^s$ EAF, EBF.

$\because$ { two $\angle^s$ of the one = two $\angle^s$ of the other, each to each,
and the side EF, w^h is opp. to equal $\angle^s$ in each $\triangle$, is com. to both; }

$\therefore$ side AF = side BF. 26. 1.

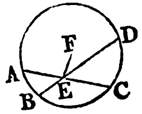
$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. IV. THEOR.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

In ⊙ ABCD, let two $|^s$ AC, BD, w^h do not both pass through the cent. cut each other in p^t E: they shall not bis^t each other.

For, if it be possible, let AE = EC, BE = ED.



If one of the $|^s$ pass through the cent. it is plain that it cannot be bis^d by the other w^h does not pass through the cent.

But, if neither of the $|^s$ pass through the cent.

1.3. find F the cent. of the ⊙, and join FE: then

∵ | FE, passing through the cent. bis^{ts} | AC,

Hyp. w^h does not pass through the cent.

∴ FE cuts AC at r^t ∠^{s};

3.3. and ∴ ∠ AEF is a r^t ∠.

Again,

Hyp. ∵ | FE bis^{ts} | BD, w^h does not pass through the cent.

∴ FE cuts BD at r^t ∠^{s};

3.3. and ∴ ∠ FEB is a r^t ∠:

but, from above,

∠ AEF is also a r^t ∠;

∴ ∠ AEF = ∠ BEF,

Ax. 1. *i. e.* the less = the greater,

w^h is impossible:

∴ AC, BD do not bis^t each other.

[Q. E. D.]

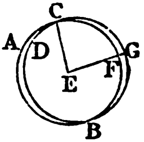
∴ *if in a circle, &c.*

PROP. V. THEOR.

If two circles cut one another, they shall not have the same centre.

Let the ⊙s ABC, CDG cut each other in the pts B, C: they shall not have the same cent.

For, if it be possible, let E be their common cent.: join EC, and draw any | EFG meeting the ⊙s in F and G: then,



∵ E is cent. of ⊙ ABC,

∴ EC = EF: Def. 15. 1.

Again,

∵ E is cent. of ⊙ CDG,

∴ EC = EG: Def. 15. 1.

but EC = EF;

∴ EF = EG, Ax. 1.

i. e. the less = the greater;

wh is impossible.

∴ E is not the cent. of the ⊙s ABC, CDG.

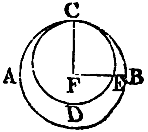
∴ *if two circles, &c.* [Q. E. D.]

PROP. VI. THEOR.

If two circles touch one another internally, they shall not have the same centre.

Let the ⊙s ABC, CDE touch each other internally in the pt C: they shall not have the same cent.

For, if they have, let this cent. be F: join FC, and draw any | FEB meeting the ⊙s in E and B: then,



$\because$ F is cent. of ⊙ ABC,

Def. 15. $\therefore$ FC = FB :

1. also, $\because$ F is cent. of ⊙ CDE,

$\therefore$ FC = FE :

but FC = FB ;

Ax. 1. $\therefore$ FE = FB,

or the less = the greater

wh is impossible :

$\therefore$ F is not the cent. of the ⊙s ABC, CDE.

$\therefore$ *if two circles, &c.* [Q. E. D.]

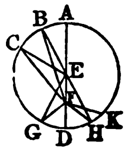
PROP. VII. THEOR.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: and from the same point there can be drawn only two straight lines that are equal to one another, one upon each side of the shortest line.

Let ABCD be a ⊙, AD its diamr, E its cent. and in AD let any pt F be taken wh is not the cent. of all the |s FB, FC, FG, &c. that can be drawn from F to the ⊙ce, FA shall be the greatest, and FD, the other part of the diamr AD, the least: and of the others, FB shall be > FC, FC > FG, &c.

Join BE, CE, GE: then,

$\because$ two sides of a $\triangle$ are $>$ the third, 20. 1.



$\therefore$ BE + EF $>$ BF:

but AE = BE; Def. 15 1.

$\therefore$ AE + EF $>$ BF,

i.e. AF $>$ BF:

Again, in $\triangle^s$ BEF, CEF,

$\because \begin{cases} \text{BE} = \text{CE, FE is com. to both,} \\ \text{but } \angle \text{ BEF} > \text{CEF;} \end{cases}$ Ax. 9. 1

$\therefore$ base BF $>$ base CF: 24. 1.

and for the same reason,

CF $>$ GF:

Again, $\because$ GF + FE $>$ EG. 20. 1.

and that EG = ED,

$\therefore$ GF + FE $>$ ED:

take away the com. part FE;

then, rem^r GF $>$ FD. Ax. 5.

$\therefore$ FA is the greatest, and FD is the least, of all the $|^s$ drawn from F to the $\odot^{ce}$; and BF is $>$ CF, CF $>$ GF.

Also, there can be drawn only two equal $|^s$ from the p^t F to the $\odot^{ce}$, one on each side of the shortest $|$ FD.

At p^t E in $|$ EF, make $\angle$ FEH = $\angle$ FEG, 23. 1. and join FH.

then, in $\triangle^s$ GEF, HEF,

$\because \begin{cases} \text{side GE} = \text{EH, and EF is com. to both,} \\ \text{and } \angle \text{ GEF} = \angle \text{ HEF;} \end{cases}$ Def. 15. 1. Constr

$\therefore$ base FG = base FH: 4. 1.

but besides this $|$ FH, no other $|$ can be drawn from F to the $\odot^{ce}$ that shall be = FG:

for, if there can, let it be $|$ FK:

then, $\because$ $FK = FG$,
and $FG = FH$,
Ax. 1. $\therefore$ $FK = FH$,
i. e. a | nearer to that w[h] passes through the cent. is = one w[h] is more remote;
but it has been shown that this is impossible.

$\therefore$ *if any point be taken, &c.* [Q. E. D.]

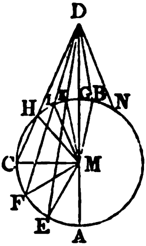
PROP. VIII. THEOR.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote: but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote: and only two equal straight lines can be drawn from the same point to the circumference, one upon each side of the least line.

Let ABC be a ⊙, and D any p[t] without it, from w[h] let |[s] DA, DE, DF, DC, be drawn to the ⊙[ce], whereof DA passes through the cent.

Of those w[h] fall on the concave part of the ⊙[ce] AEFC, the greatest shall be DA, w[h] passes

through the cent. and the nearer to it shall be $>$ those more remote, viz. DE $>$ DF, DF $>$ DC: but of those w[h] fall on the convex $\odot$[ce] HLKG, the least shall be DG between p[t] D and the diam[r] AG; and the nearer to it shall be $<$ those more remote, viz. DK $<$ DL, DL $<$ DH.



Find the cent. M of $\odot$ ABC, 1. 3.
and join ME, MF, MC, MK, ML, MH.
Then, AM = ME;
add MD;
$\therefore$ AD = ME + MD: Ax. 2.
but (ME + MD) $>$ ED: 20. 1.
$\therefore$ also AD $>$ ED.
Again, in $\triangle$s EMD, FMD,
$\because \begin{cases} \text{side ME = MF, MD is com. to both,} \\ \text{but } \angle \text{ EMD} > \angle \text{ FMD;} \end{cases}$ Ax. 9.
$\therefore$ base ED $>$ base FD: 24. 1.
in like manner it may be shown, that
FD $>$ CD.
And
$\therefore$ DA is the greatest |; DE $>$ DF, DF $>$ DC.
Also $\because$ (MK + KD) $>$ MD, 20. 1.
and MK = MG, Def. 15. 1.
$\therefore$ rem[r] KD $>$ rem[r] GD, Ax. 5.
i.e. GD $<$ KD:
and
$\because$ from the ext[ics] M, D of the side MD of $\triangle$MLD,
the |[s] MK, DK are drawn to p[t] K within the $\triangle$,
$\therefore$ (MK + DK) $<$ (ML + LD): 21. 1.

Def. 15. but $MK = ML$;
1. Ax. 5. $\therefore$ the rem^r $DK < $ the rem^r DL:
in like manner it may be shown, that
$DL < DH$.

And
$\therefore$ DG is the least |; $DK < DL$, $DL < DH$.

Also there can be drawn only two equal |^s from the p^t D to the ⊙^ce, one on each side of the least |.

23. 1. At p^t M, in | MD, make $\angle DMB = \angle DMK$,
and join DB:
then, in $\triangle^s$ KMD, BMD,
$\because$ { side $MK = MB$, MD is com. to both,
Constr. and $\angle KMD = \angle BMD$;
4. 1. $\therefore$ base $DK =$ base DB:
but, besides this | DB, no other | can be drawn from D to the ⊙^ce that shall be $= DK$:
for, if there can, let it be DN:
then, $\because$ $DK = DN$,
and also $DK = DB$,
$\therefore$ $DB = DN$,
i. e. a | nearer to the least | is = one more remote, w^h has been proved to be impossible.

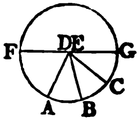
$\therefore$ *if any point, &c.* [Q. E. D.]

PROP. IX. THEOR.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Within the ⊙ ABC let p^t D be taken, from w^h to the ⊙^ce there fall more than two equal |^s, viz. DA, DB, DC: D shall be the cent. of the ⊙.

For, if not, let E be the cent.; join DE, and prod. it to the ⊙ ^{ce} in F, G; then, FG is a diam^r of the ⊙ ABC: Def. 17



and

∵ in FG, the diam^r of ⊙ ABC, there is taken the p^t D, w^h is not the cent.

∴ DG is the greatest | from it to the ⊙ ^{ce}, 7. 3.
and DC > DB, DB > DA:

but these |^s are also = one another, Hyp.
w^h is impossible:

∴ E is not the cent. of ⊙ ABC:

And in like manner it may be shown, that no other p^t but D is the cent.

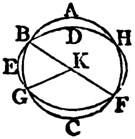
∴ D is the cent.

∴ *if a point be taken, &c.* [Q. E. D.]

PROP. X. THEOR.

One circumference of a circle cannot cut another in more than two points.

If possible, let the ⊙ ^{ce} FAB cut the ⊙ ^{ce} DEF in more than two p^{ts}, viz. in B, G, F.



Find the cent. K of ⊙ ABC, and join KB, KG, KF: 3. 3.

Then ∵ K is cent. of ⊙ ABC, Def. 15.
∴ KB = KG = KF: 1.

and ∵ within $\odot$ DEF there is taken the p^t K, from w^h to the $\odot^{ce}$ DEF fall more than two equal |^s KB, KG, KF,

9. 3. ∴ K is the cent. of $\odot$ DEF:

Constr. but K is also the cent. of $\odot$ ABD;

and ∴ the same p^t is the cent. of two $\odot^s$ w^h cut one another;

5. 3. but this is impossible.

∴ *one circumference, &c.* [Q. E. D.]

PROP. XI. THEOR.

If two circles touch each other internally, the straight line which joins their centres being produced shall pass through the point of contact.

Let the two $\odot^s$ ABC, ADE touch each other internally in the p^t A; and let F be the cent. of $\odot$ ABC, G the cent. of $\odot$ ADE: the | w^h joins the cent^s F, G, being prod^d, shall pass through the p^t A.

For, if not, let it, if possible, fall otherwise, as FGDH, and join AF, AG.

Then,

20. 1. ∵ two sides of a △ are > the third,

∴ $(FG + AG) > FA$:

Def. 5. 1. but $FA = FH$;

∴ $(FG + AG) > FH$:

take away the com. part FG;

Ax. 5. ∴ the rem^r $AG >$ the rem^r GH:

Def. 15. 1. but $AG = GD$;

and $\therefore$ GD > GH,

or the less > the greater,

wh is impossible.

$\therefore$ the | wh joins the pts F, G, being prodd, cannot fall otherwise than on the pt A,

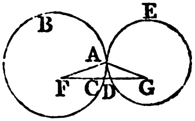
i. e. it must pass through A.

$\therefore$ *if two circles, &c.* [Q. E. D.]

PROP. XII. THEOR.

If two circles touch each other externally, the straight line which joins their centres, shall pass through the point of contact.

Let the ⊙s ABC, ADE touch each other externally in the pt A; and let F be the cent. of ⊙ ABC, G the cent. of ⊙ ADE: the | wh joins the pts F, G, shall pass through the pt of contact A.



For, if not, let it, if possible, fall otherwise, as FCDG, and join FA, AG.

Then, $\because$ F is cent. of ⊙ ABC,

$\therefore$ FA = FC:

also, $\because$ G is cent. of ⊙ ADE,

$\therefore$ GA = GD,

$\therefore$ (FA + AG) = (FC + DG); Ax. 2.

and $\therefore$ (FA + AG) < the whole FG:

but (FA + AG) > the same FG; 20 1.

i. e. (FA + AG) is both > and < FG;

wh is impossible;

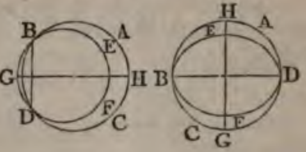
∴ the | w^h joins the p^{ts} F, G cannot pass otherwise than through the p^t of contact A,
i. e. it must pass through it.

∴ *if two circles, &c.* [Q. E. D.]

PROP. XIII. THEOR.

One circle cannot touch another in more points than one, whether it touches it on the inside or outside.

For, if possible, let ⊙ EBF touch ⊙ ABC in more p^{ts} than one; and first, on the inside, in the
10.11.1. p^{ts} B, D: join BD, and draw GH, bis^g BD at r^t $\angle^s$:

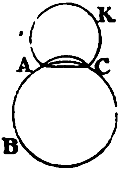


Then,

∵ the p^{ts} B, D are in the ⊙ ^{ces} of each ⊙,
2. 3. ∴ | BD falls within each ⊙;
Cor. 1. 3. ∴ the $cent^s$ of both ⊙^{s} are in | GH,
w^h $bist^s$ BD at r^t $\angle^s$;
11. 3. and ∴ GH passes through the p^t of contact:
But ∵ the p^{ts} B, D are both without | GH,
∴ GH does not pass through this p^t,
i. e. GH does and does *not* pass through the same p^t,
w^h is absurd:

∴ one ⊙ cannot touch another on the inside in more p^{ts} than one.

Nor can two $\odot$ˢ touch one another on the outside in more than one pᵗ.



For, if possible, let $\odot$ ACK touch $\odot$ ABC in pᵗˢ A, C : join AC, then,

∵ the two pᵗˢ A, C are in the $\odot$ᶜᶜ of $\odot$ ACK,

∴ | AC falls within $\odot$ ACK : 2. 3.

but $\odot$ ACK is without $\odot$ ABC ; Hyp.

∴ | AC is without this $\odot$ ABC :

but ∵ pᵗˢ A, C are in the $\odot$ᶜᶜ of $\odot$ ABC,

∴ | AC must be within $\odot$ ABC, 2. 3.

i. e. AC is both within and without the same $\odot$, wʰ is absurd :

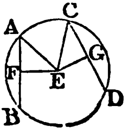
∴ one $\odot$ cannot touch another on the outside in more pᵗˢ than one.

∴ *one circle, &c.* [Q. E. D.]

PROP. XIV. THEOR.

Equal straight lines in a circle are equally distant from the centre ; and those which are equally distant from the centre, are equal to one another.

In $\odot$ ABDC, let | AB = | CD ; they shall be equally distant from the cent.



Find the cent. E of $\odot$ ABDC, 1. 3.
from it draw EF, EG, ⊥ AB, CD, 12. 1.
and join EA, EC.

Then,

$\because$ | EF cuts | AB at $r^t \angle^s$,

3. 3. $\therefore$ EF bis^{ts} AB:

$\therefore$ $AF = BF$,

and $AB = 2\ AF$:

for the same reason,

$CD = 2\ CG$:

Hyp. but $AB = CD$;

Ax. 7. $\therefore$ $AF = CG$,

and $AF^2 = CG^2$.

Def. 15. 1. And $\because$ $AE = EC$,

$\therefore$ $AE^2 = EC^2$:

but $\because$ AFE, EGC are $r^t \angle^s$,

47. 1. $\therefore$ $AE^2 = AF^2 + FE^2$,

and $EC^2 = EG^2 + GC^2$:

Ax. 1. $\therefore$ $AF^2 + FE^2 = EG^2 + GC^2$:

but, from above, $AF^2 = CG^2$,

Ax. 3. $\therefore$ rem^r $EF^2 =$ rem^r EG^2,

and $EF = EG$:

4 Def. 3. but |^s in a ⊙ are said to be equally distant from the cent. when the $\perp^s$ let fall upon them from the cent. are = one another:

$\therefore$ |^s AB, CD are equally distant from the centre.

Next, let |^s AB, CD be equally distant from the cent. *i. e.* let $EF = EG$: then $AB = CD$.

For, the same constr^n being made, it may be shown, as before, that

$AB = 2\ AF$,

$CD = 2\ CG$;

and also, $EF^2 + AF^2 = EG^2 + CG^2$.

but $\because$ $EF = EG$, Hyp.

$\therefore$ $EF^2 = EG^2$;

$\therefore$ rem^r $AF^2 =$ rem^r CG^2, Ax. 3.

and $\therefore$ $AF = CG$:

but $AB = 2\,AF$,

$CD = 2\,CG$,

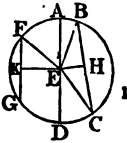
and $\therefore$ $AB = CD$. Ax. 6.

$\therefore$ *equal straight lines, &c.* [Q. E. D.]

PROP. XV. THEOR.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let ABCD be a $\odot$, E its cent. AD a diam^r; and let BC be nearer to the cent. than FG: AD shall be $>$ any | BC, w^h is not a diam^r; and BC $>$ FG.



From cent. E draw EH, EK $\perp^s$ to BC, FG; and join EB, EC, EF. 12. 1.

Then, $\because$ $AE = EB$,

and $ED = EC$, Def. 15. 1.

$\therefore$ $AD = (EB + EC)$: Ax. 2.

but $(EB + EC) > BC$; 20. 1

and $\therefore$ $AD > BC$.

Again,

$\because$ BC is nearer to the cent. than FG, Hyp.

$\therefore$ $EH < EK$ Def. 5. 3.

and $EH^2 < EK^2$:

but, as was shown in the preceding propⁿ,

$$BC = 2\,BH,$$
$$FG = 2\,FK,$$
$$\text{and } EH^2 + BH^2 = EK^2 + FK^2:$$
$$\text{but, from above, } EH^2 < EK^2:$$
$$\therefore\ BH^2 > FK^2,$$
$$\text{and } BH > FK;$$
$$\therefore\ BC > FG.$$

Next, let BC be $>$ FG: BC shall be nearer to
Def.5.3. the cent. than FG, *i. e.* the same constrⁿ being made, EH shall be $<$ EK.

$$\text{For, } \because\ BC > FG,$$
$$\therefore\ BH > FK;$$
$$\text{and } BH^2 > FK^2:$$
$$\text{but } BH^2 + EH^2 = FK^2 + EK^2;$$
$$\therefore\ EH^2 < EK^2,$$
$$\text{and } EH < EK;$$

Def.5.3. and $\therefore$ BC is nearer to the cent. than FG.

$\therefore$ *the diameter, &c.* [Q. E. D.]

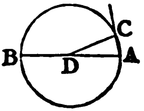
PROP. XVI. THEOR.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight

line which is at right angles to it, as not to cut the circle.

Let ABC be a ⊙, D its cent. and AB a diam^r: the | drawn at r^t ∠^{s} to AB, from its extr^y A, shall fall without the ⊙.

For, if not, let it, if possible, fall within the ⊙, as AC; and draw DC to the p^t C, in w^h it meets the ⊙^{ce}.



Then,

∵	DA = DC,	Def. 15 1.
∴	∠ DAC = ∠ DCA:	5. 1.
but	DAC is a r^t ∠;	Hyp.
∴	DCA is also a r^t ∠;	
and ∴	∠^{s} (DAC + ACD) = two r^t ∠^{s}:	
	but this is impossible:	32. 1.

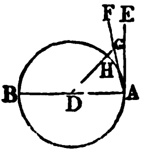
∴ the | drawn from A at r^t ∠^{s} to BA does not fall within the ⊙:

and in the same manner it may be shown, that it does not fall upon the ⊙^{ce}.

∴ it must fall without the circle, as AE.

Also, between the | AE and the ⊙^{cc} no other | can be drawn from the p^t A w^h does not cut the ⊙

For, if possible, let AF be between them: from the p^t D draw DG ⊥ to AF, and let it meet the ⊙^{ce} in H. 12. 1.



Then, ∵ AGD is a r^t ∠,

and DAG < a r^t ∠, 17. 1.

∴ DA > DG: 19. 1.

Def. 15. but $DA = DH$;

1. $\therefore$ $DH > DG$,

i.e. the less > the greater,

wh is impossible.

$\therefore$ *no straight line can be drawn from the point* A *between* AE *and the circumference, which does not cut the circle: or, which amounts to the same thing, however great an acute angle a straight line makes with the diameter at the point* A, *or however small an angle it makes with* AE, *the circumference must pass between that straight line and the perpendicular* AE.

"And this is all that is to be understood, when, in the Greek text, and translations from it, the $\angle$ of the $\frac{1}{2}$ $\odot$ is said to be > any acute rectl $\angle$, and the remg $\angle$ < any rectl $\angle$." [Q. E. D.]

Cor.—From this it is manifest, that the | wh is drawn at rt $\angle$s to the diamr of a $\odot$, from the extry

Def. 2. 3. of it, touches the $\odot$; and that it touches it only in one pt; for, if it did meet the $\odot$ in two, it would

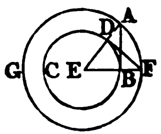
2. 3. fall within it. "Also it is evident, that there can be but one | wh touches the $\odot$ in the same pt."

PROP. XVII. PROB.

To draw a straight line from a given point either without or in the circumference, which shall touch a given circle.

First, let the given pt A be without the given $\odot$ BCD: it is reqd to draw from A a | wh shall touch the $\odot$.

Find the cent. E of the $\odot$; 1. 3.
join AE; and from cent. E,
at dist. EA, desc. $\odot$ AFG;
from p^t D draw DF at $r^t \angle^s$ 11. 1.
to EA, and join EBF, AB:
AB shall touch the $\odot$ BCD.



For, $\because$ E is cent. of $\odot^s$ BCD, AFG, Def. 15. 1.

$\therefore$ EA = EF, ED = EB:

Hence, in $\triangle^s$ AEB, FED,

$\because \begin{cases} \text{side } AE = EF,\ EB = ED, \\ \text{and } \angle\ AEB \text{ is com. to the two } \triangle^s, \end{cases}$

$\therefore \begin{cases} \text{the base } DF = \text{the base } AB, \\ \triangle\ FED = \triangle\ AEB, \\ \text{and the other } \angle^s = \text{the other } \angle^s: \end{cases}$ 4. 1.

$\therefore$ $\angle$ EBA = $\angle$ EDF:

but EDF is a $r^t \angle$; Constr.

$\therefore$ EBA is a $r^t \angle$: Ax. 1.

and EB is drawn from the cent.

but a | drawn from the $extr^v$ of a $diam^r$,
at $r^t \angle^s$ to it, touches the $\odot$. Cor. 16. 1.

$\therefore$ AB *touches the circle; and it is drawn from the given point* A. [Q. E. F.]

But if the given p^t shall be in the $\odot^{ce}$ of the $\odot$, as the p^t D, draw DE to the cent. E, and DF at $r^t \angle^s$ to DE: DF touches the $\odot$. Cor. 16. 3.

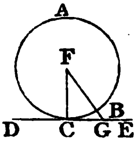
PROP. XVIII. THEOR.

If a straight line touches a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Let the | DE touch the $\odot$ ABC in the p^t C;

and from the cent. F, let | FC be drawn: FC shall be ⊥ to DE.

1. 3. For, if not, from the p[t] F draw FG ⊥ to DE: then,



17. 1. ∵ FGC is a r[t] ∠,
∴ GCF is an acute ∠;
19. 1. and to the greater ∠ the greater side is opp.
∴ FC > FG:
Def. 15. 1. but FC = FB:
∴ FB > FG,
i. e. the less > the greater,
w[h] is impossible:
∴ FG is not ⊥ to DE.

And in the same manner it may be shown, that no other | is ⊥ to DE, but FC: *i.e.* FC is ⊥ to DE.

∴ *if a straight line, &c.* [Q. E. D.]

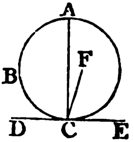
PROP. XIX. THEOR.

If a straight line touches a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the | DE touch the ⊙ ABC in p[t] C, and from C let CA be drawn at r[t] ∠[s] to DE: the cent. of the ⊙ shall be in CA.

For, if not, let, if possible, F be the cent. and join CF: then,

∵ DE touches the ⊙ ABC,
and FC is drawn from the cent. to the p[t] of contact,
∴ FC is ⊥ to DE, 18. 3.
and ∴ FCE is a r[t] ∠ :
but ACE is also a r[t] ∠ ; Hyp.
and ∴ ∠ FCE = ∠ ACE, Ax. 1
i.e. the less = the greater,
w[h] is impossible:



∴ F is not the cent. of ⊙ ABC.

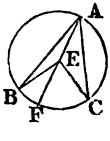
And in the same manner it may be shown, that no other p[t], w[h] is not in CA, is the cent.
i.e. the cent. is in CA.

∴ *if a straight line, &c.* [Q. E. D.]

PROP. XX. THEOR.

The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

In ⊙ ABC, let BEC be an ∠ at the cent. BAC an ∠ at the ⊙[ce], w[h] have the same arc BC for their base:
∠ BEC shall be double of ∠ BAC.

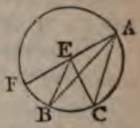


Join AE, and prod. it to F; and, first, let the cent. of the ⊙ be within the ∠ BAC.

Then ∵ EA = EB,
∴ ∠ EAB = ∠ EBA ; 5. 1.
and ∴ ∠[s] (EAB + EBA) = 2 ∠ EAB;
but ∠[s] (EAB + EBA) = ∠ BEF: 32. 1.

and ∴ $\angle$ BEF $= 2 \angle$ EAB:
for the same reason, $\angle$ FEC $= 2 \angle$ EAC:
and
∴ the whole $\angle$ BEC is double of the whole $\angle$ BAC.

Again, let the cent. of the ⊙ be without the $\angle$ BAC: then it may be shown, as in the first case, that



$\angle$ FEC is double of $\angle$ FAC
and that
part FEB is double of part FAB;
∴ rem^g $\angle$ BEC is double of the rem^g $\angle$ BAC.
∴ *the angle at the centre, &c.* [Q.E.D.].

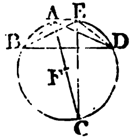
PROP. XXI. THEOR.

The angles in the same segment of a circle are equal to one another.

In ⊙ ABCD, let BAD, BED be $\angle^s$ in the same seg^t BAED: then shall
$\angle$ BAD $= \angle$ BED.

First, let seg^t BAED be $> \frac{1}{2}$ ⊙:
1. 3. take the cent. F of ⊙ ABCD, and join FB, FD.

Then, ∵ $\angle$ BFD is at the cent.
$\angle$ BAD at the ⊙^{ce},
and these $\angle^s$ have the same arc BCD for their base,
20. 3. ∴ $\angle$ BFD $= 2 \angle$ BAD:
for the same reason,
$\angle$ BFD $= 2 \angle$ BED:
Ax. 7. and ∴ $\angle$ BAD $= \angle$ BED.



Next, if seg^t BAED be $< \frac{1}{2} \odot$, draw AF to the cent. prod. it to C, and join CE:

then, seg^t BADC $> \frac{1}{2} \odot$, and $\therefore$, by the first case,

$\angle$ BAC $= \angle$ BEC:

similarly, $\because$ CBED $> \frac{1}{2} \odot$,

$\therefore \angle$ CAD $= \angle$ CED:

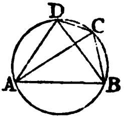
and $\therefore$ the whole $\angle$ BAD $=$ the whole $\angle$ BED. Ax. 2.

$\therefore$ *the angles in the same segment, &c.*

[Q. E. D.]

PROP. XXII. PROB.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilat^l fig. inscr^d in $\odot$ ABCD: any two of its opp. $\angle^s$ shall together be $=$ two r^t $\angle^s$

Join AC, BD: then,

$\because$ the $\angle^s$ in the same seg^t are $=$ one another, 21 3

$\therefore$ $\angle$ CAB $= \angle$ CDB in the seg^t CDAB,

also, $\angle$ ACB $= \angle$ ADB in the seg^t ADCB,

and $\therefore \angle^s$ (CAB $+$ ACB) $=$ the whole $\angle$ ADC. Ax. 2

add $\angle$ ABC: then,

$\angle^s$(CAB $+$ ACB $+$ ABC) $= \angle^s$(ADC $+$ ABC): Ax. 2

but,

CAB, ABC, ACB are the three $\angle^s$ of $\triangle$ CAB,

and $\therefore \angle^s$(CAB $+$ ABC $+$ ACB) $=$ two r^t $\angle^s$; 32. 1.

$\therefore$ also, $\angle^s$(ADC $+$ ABC) $=$ two r^t $\angle^s$. Ax. 1

And in the same manner it may be shown that

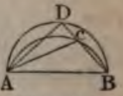
$\angle^s(BAD + DCB) =$ two $r^t \angle^s$.

∴ *the opposite angles, &c.* [Q. E. D.]

PROP. XXIII. THEOR.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If possible, on the same | AB, and on the same side of it, let there be two sim^r seg^ts of ⊙^s, ACB, ABD, not coinciding with one another.



Then,

∵ ⊙ ACB cuts ⊙ ADB in the two p^ts A, B,

10. 3. ∴ they cannot cut one another in any other p^t: and ∴ one of the seg^ts must fall within the other: let ACB fall within ADB: draw | BCD, and join CA, DA;

Then,

Hyp. ∵ seg^t ACB is sim^r to seg^t ADB,

Def. 11. 3. and that sim^r seg^ts contain equal $\angle^s$;

∴ $\angle ACB = \angle ADB$,

i. e. the ext^r $\angle$ = the int^r $\angle$,

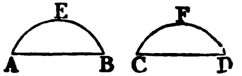
16. 1 w^h is impossible.

∴ *there cannot be two similar segments of circles on the same side of the same straight line, which do not coincide.* [Q. E. D.]

PROP. XXIV. THEOR.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD be sim^r seg^{ts} of $\odot^s$ on equal $|^s$ AB, CD: the seg^t AEB shall be = the seg^t CFD.



For, let the seg^t AEB be applied to seg^t CFD, so that the p^t A may be on C, and | AB on CD:

then, $\because$ AB = CD,

$\therefore$ p^t B shall coincide with p^t D:

Hence, | AB coinciding with CD,

the seg^t AEB must coincide with CFD, 23. a

and $\therefore$ seg^t AEB = seg^t CFD. Ax. 8.

$\therefore$ *similar segments, &c.* [Q. E. D.]

PROP. XXV. PROB.

A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given seg^t of a $\odot$: it is req^d to desc. the $\odot$ of w^h it is the seg^t.

Bis^t AC in D, from D draw DB at r^t $\angle^s$ to AC, 10. 11. I.
and *join AB*.

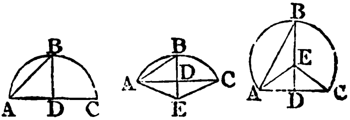


Fig. 1. First, let $\angle$ ABD $=$ $\angle$ BAD:
6. 1. then, $|$ BD $=$ $|$ DA;
but DA $=$ DC;
$\therefore$ DB $=$ DA $=$ DC;
9. 3. and $\therefore$ D is the cent. of the $\odot$.

From the cent. D, at dist. DA, DB, or DC, desc. a $\odot$; it shall pass through the other two p[ts], and be the $\odot$ req[d]:
and $\because$ the cent. D is in AC,
$\therefore$ the seg[t] ABC is a $\frac{1}{2}$ $\odot$

Figs. 2. 3 But, if $\angle$ ABD $\not=$ BAD;
23. 1. at p[t] A, in $|$ AB, make $\angle$ BAE $=$ ABD;
prod. BD, if necessary, to E, and join EC.

Then, $\because$ $\angle$ ABE $=$ $\angle$ BAE,
6. 1. $\therefore$ $|$ BE $=$ AE:
also, in $\triangle$s ADE, CDE,

Constr $\because$ $\begin{cases} \text{side AD} = \text{DC}, \\ \text{DE is com. to both}, \\ \text{and r}^{\text{t}} \angle \text{ ADE} = \text{r}^{\text{t}} \angle \text{ CDE}, \end{cases}$

4. 1. $\therefore$ the base AE $=$ the base EC:
and, from above, AE $=$ BE:
Ax. 1. $\therefore$ EA $=$ EB $=$ EC;
9. 3. and $\therefore$ E is the cent. of the $\odot$.

From the cent. E, at dist. EA, EB or EC, desc. a $\odot$: it shall pass through the other p[ts], and be *the $\odot$ req[d].*

And it is evident, that

if $\angle$ ABD be $>$ $\angle$ BAD, Fig. 2.

the cent. E falls without the seg^t ABC,

and $\therefore$ the seg^t is $<$ a $\frac{1}{2}$ $\odot$:

but, if $\angle$ ABD be $<$ $\angle$ BAD, Fig. 3.

the cent. E falls within the seg^t ABC,

and $\therefore$ this seg^t is $>$ a $\frac{1}{2}$ $\odot$.

Hence, *a segment of a circle being given, the circle is described of which it is the segment.*

[Q. E. F.]

PROP. XXVI. THEOR.

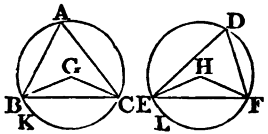
In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

Let $\odot$ ABC $=$ $\odot$ DEF ; and

let $\angle$ BGC $=$ $\angle$ EHF at the cent^s of these $\odot^s$,

and $\angle$ BAC $=$ $\angle$ EDF at the $\odot^{ces}$ of the $\odot^s$:

the arc BKC shall be $=$ the arc ELF :



Join BC, EF : then,

$\because$ $\odot$ ABC $=$ $\odot$ DEF,

$\therefore$ the radii of one $\odot$ $=$ the radii of the other : Def. 1. 3.

hence, in the two $\triangle^s$ BGC, EHF,

$\because$ $\begin{cases} \text{side GB} = \text{HE, GC} = \text{HF,} \\ \text{and also, } \angle \text{ at G} = \angle \text{ at H ;} \end{cases}$ Hyp.

$\therefore$ the base BC $=$ the base EF. 4. 1.

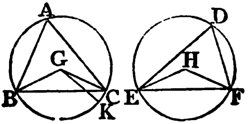
Hyp. And $\because$ the $\angle$ at A = the $\angle$ at D,
Def. 11. 3. $\therefore$ the seg^t BAC is sim^r to the seg^t EDF;
and they are on equal |^s BC, EF:
24. 3. but sim^r seg^{ts} of $\odot^s$ on equal |^s are = one another;
and $\therefore$ seg^t BAC = seg^t EDF:
Hyp. but the whole $\odot$ ABC = the whole $\odot$ DEF;
Ax. 3. $\therefore$ the rem^g seg^t BKC = the rem^g seg^t ELF;
and the arc BKC = the arc ELF.

$\therefore$ *in equal circles, &c.* [Q. E. D.]

PROP. XXVII. THEOR.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let $\odot$ ABC = $\odot$ DEF; and let the $\angle^s$ BGC, EHF at their cent^s, and BAC, EDF at their $\odot^{ces}$, stand on equal arcs BC, EF: then shall $\angle$ BGC = $\angle$ EHF, and $\angle$ BAC = $\angle$ EDF.



If $\angle$ BGC = $\angle$ EHF,
20. 3. & Ax. 7. 1. it follows that $\angle$ BAC = $\angle$ EDF.
But, if $\angle$ BGC $\not\equiv$ EHF,
one must be > the other:
let BGC be > EHF,
23. 1. and at the p^t G, in | BG, make $\angle$ BGK = $\angle$ EHF:

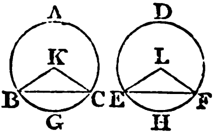
then, $\because \angle$ BGK $= \angle$ EHF,
and that equal $\angle$s at the cents stand on equal arcs, 26. 3.
$\therefore$ arc BK $=$ arc EF:
but arc EF $=$ arc BC; Hyp.
$\therefore$ also BK $=$ BC, Ax. 1.
i. e. the less $=$ the greater,
wh is impossible:
$\therefore \angle$ BGC is not $\neq \angle$ EHF,
i. e. $\angle$ BGC $= \angle$ EHF:
and the $\angle$ at A $= \frac{1}{2} \angle$ BGC, 20. 3.
the $\angle$ at D $= \frac{1}{2} \angle$ EHF;
$\therefore$ the $\angle$ at A $=$ the $\angle$ at D. Ax. 7.

$\therefore$ *in equal circles, &c.* [Q. E. D.]

PROP. XXVIII. THEOR.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Let $\odot$ ABC $=$ $\odot$ DEF, and in these $\odot$s let $|$ BC $= |$ EF, these $|$s cutting off the two greater arcs BAC, EDF, and the two less BGC, EHF: then shall arc BAC $=$ EDF, and BGC $=$ EHF.



Take K, L, the cents of the $\odot$s, and join 1. 3. BK, KC EL, LF: then,

$\because \odot ABC = \odot DEF$,

Def. 13. $\therefore$ the radii of one $\odot$ = the radii of the other:
hence, in $\triangle^s$ BKC, ELF,

side $BK = EL$, $CK = FL$,

Hyp. also, the base BC = the base EF;

8. 1. and $\therefore \angle BKC = \angle ELF$:

26. 3. but equal $\angle^s$ at the cent^s stand on equal arcs;

$\therefore$ arc BGC = arc EHF:

Hyp. and the whole $\odot^{ce}$ ABC = the whole $\odot^{ce}$ EDF;

Ax. 3. $\therefore$ the rem^g arc BAC = the rem^g arc EDF.

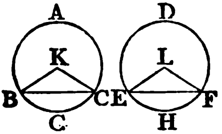
$\therefore$ *in equal circles, &c.* [Q. E. D.]

PROP. XXIX. THEOR.

In equal circles, equal circumferences are subtended by equal straight lines.

Let $\odot ABC = \odot DEF$; also, let arc BGC = arc EHF; and join BC, EF:

$|BC$ shall be $= |EF$.



1. 3. Take K, L, the cent^s of the $\odot^s$, and join BK, KC, EL, LF:

then, $\because$ arc BGC = arc EHF,

27. 3. $\therefore \angle BKC = \angle ELF$:

and $\because \odot ABC = \odot DEF$,

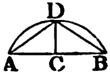
Def. 13. $\therefore$ the radii of one $\odot$ = the radii of the other:

hence, in the $\triangle^s$ BKC, ELF,
side $BK = EL$, $KC = LF$,
also $\angle BKC = \angle ELF$;
and $\therefore$ the base $BC =$ the base EF. 4. 1.
$\therefore$ *in equal circles, &c.* [Q. E. D.]

PROP. XXX. PROB.

To bisect a given circumference, that is, to divide it into two equal parts.

Let ADB be the given arc: it is req^d to bis^t it.
Join AB, and bis^t it in the p^t C: 10. 1.
from C draw CD at r^t $\angle^s$ to AB: 11. 1.
the arc ADB shall be bis^d in D.



Join AD, DB:
then, in $\triangle^s$ ACD, BCD,

$\because \begin{cases} \text{side } AC = CB, \\ CD \text{ is com. to both,} \\ \text{and r}^t \angle ACD = \text{r}^t \angle BCD, \end{cases}$

$\therefore$ the base $AD =$ the base BD 4. 1.
But equal $|^s$ cut off equal arcs, 28. 3.
the greater = the greater, the less = the less;
and $\because$ DC passes through the cent., Cor. 1. 3.
AD, BD are each $<$ a $\frac{1}{2}$ $\odot$;
$\therefore$ the arc $AD =$ the arc DB.
$\therefore$ *the given arc is bisected in* D. [Q. E. F.]

PROP. XXXI. THEOR.

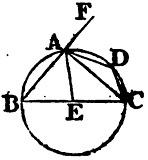
In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than

a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a ⊙, E its cent., BC a diam^r; draw CA, div^g the ⊙ into the seg^ts ABC, ADC; and join BA, AD, DC.

then the $\angle$ in the $\frac{1}{2}$ ⊙ BAC shall be a r^t $\angle$;
the $\angle$ in the seg^t ABC, w^h is $>$ $\frac{1}{2}$ ⊙, shall be $<$ a r^t $\angle$;
the $\angle$ in the seg^t ADC, w^h is $<$ $\frac{1}{2}$ ⊙, shall be $>$ a r^t $\angle$.

Join AE, and prod. BA to F:



then,

Def. 15. 1. $\because$ EA = EB,

5. 1. $\therefore$ $\angle$ EAB = $\angle$ EBA;

also,

$\because$ EA = EC,

$\therefore$ $\angle$ EAC = ECA;

Ax. 2. and $\therefore$ $\angle$ BAC = $\angle$^s(ABC + ACB):

32. 1. but ext^r $\angle$ FAC = $\angle$^s(ABC + ACB):

Ax. 1. $\therefore$ $\angle$ BAC = $\angle$ FAC,

Def. 10. 1. and $\therefore$ each of them is a r^t $\angle$.

And $\therefore$ *the angle* BAC *in a semicircle is a right angle.*

Again, in $\triangle$ ABC,

17. 1. $\because$ the two $\angle$^s(ABC + BAC) $<$ two r^t $\angle$^s,
and that $\angle$ BAC is a r^t $\angle$,
$\therefore$ $\angle$ ABC $<$ a r^t $\angle$.

And $\therefore$ *the angle in a segment* ABC, *which is greater than a semicircle, is less than a right angle.*

And, $\because$ ABCD is a quadrilat^l fig. in a ⊙,

22. 3. $\therefore$ its two opp. $\angle$^s(ABC + ADC) = two r^t $\angle$^s,
but, from above, $\angle$ ABC $<$ a r^t $\angle$;
$\therefore$ the other $\angle$ ADC $>$ a r^t $\angle$.

And ∴ *the angle in a segment, which is less than a semicircle, is greater than a right angle.*

Besides, it is manifest, that the arc of the greater seg^t ABC falls without the r^t ∠ CAB; but the arc of the less seg^t ADC falls within the r^t ∠ CAF. "And this is all that is meant, when, in the Greek text, and the translations from it, the ∠ of the greater seg^t is said to be greater, and the ∠ of the less seg^t is said to be less, than a r^t ∠.

Cor.—From this it is manifest, that if one ∠ of a △ be = the sum of the other two, it is a r^t ∠:

for the adj^t ∠ = the same two, 32. 1.

and when the adj^t ∠^{s} = one another,

each of them is a r^t ∠. Def. 10 1.

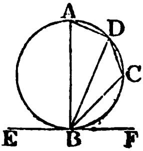
PROP. XXXII. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle; the angles which this line makes with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let | EF touch ⊙ ABCD in B, and from B let | BD be drawn, cutting the ⊙: the ∠^{s} w^h BD makes with the touching | EF shall be = the ∠^{s} in the alt. seg^{ts} of the ⊙: viz.,

∠ DBF = the ∠ in the seg^t DAB,

∠ DBE = the ∠ in the seg^t DCB.



From B draw BA at r^t ∠^{s} to EF, take any p^t C in the arc DB, and join AD, DC, CB: 11. 1.

then, $\because$ | EF touches $\odot$ ABCD in B,
and BA is drawn at r^t $\angle^s$ to EF from the p^t of contact B,

19. 3. $\therefore$ the cent. of the $\odot$ is in BA:
31. 3. $\therefore$ $\angle$ ADB in a $\frac{1}{2}$ $\odot$ is a r^t $\angle$:
32. 1. and $\therefore$ the two $\angle^s$(BAD + ABD) = a r^t $\angle$:
Constr. but $\angle$ ABF is also a r^t $\angle$:
Ax. 1. $\therefore$ $\angle$ ABF = $\angle^s$(BAD + ABD):
take away the com. part, $\angle$ ABD;
Ax. 3. then, the rem^g $\angle$ DBF = the rem^g $\angle$ BAD;
and BAD is the $\angle$ in the alt. seg^t of the $\odot$.

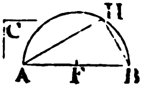
Again, $\because$ ABCD is a quadrilat^l fig. in a $\odot$,
22. 3. $\therefore$ the opp. $\angle^s$(BAD + BCD) = two r^t $\angle^s$:
13. 1. but the $\angle^s$(DBF + DBE) = two r^t $\angle^s$:
Ax. 1. $\therefore$ the $\angle^s$ (BAD + BCD) = $\angle^s$ (DBF + DBE)
and, from above,
$\angle$ BAD = $\angle$ DBF;
Ax. 2. $\therefore$ $\angle$ DBE = $\angle$ BCD,
and BCD is the $\angle$ in the alt. seg^t of the $\odot$.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. XXXIII. PROB.

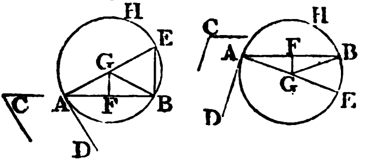
Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Let AB be the given |, and C the given $\angle$: it is req^d to desc. on AB a seg^t of a $\odot$, w^h shall contain an $\angle$ = $\angle$ C.



First, let ∠ C be a rt ∠ :
bist AB in F, and from the cent. F, 10. 1.
at the dist. FB, desc. $\frac{1}{2}$ ⊙ AHB ;
then,
the ∠ AHB in a $\frac{1}{2}$ ⊙ = the rt ∠ C. 31. 3.

But, if ∠ C be not a rt ∠ ; at pt A, in | AB, 23. 1.
make the ∠ BAD = ∠ C, and from A draw AE at 11. 1.
rt ∠s to AD ; bist AB in F, from F draw FG at 10. 1.
rt ∠s to AB, and join GB.



Then, in △s AFG, BFG,

∵ { side AF = FB, FG is com. to both,
and rt ∠ AFG = rt ∠ BFG ; Def. 10. 1.

∴ the base AG = the base BG ; 4. 1.

and ∴ the ⊙ descd from cent. G at dist. GA, shall pass through the pt B. Let AHB be this ⊙ :
the segt AHB shall contain an ∠ = the given ∠ C.

For,
∵ | AD is drawn at rt ∠s to the diamr AE from its extry A,

∴ AD touches the ⊙ : Cor. 16 3.

and ∵ | AB, drawn from the pt of contact, cuts the ⊙ ,

∴ ∠ DAB = the ∠ in the alt. segt AHB: 32. 3.

but ∠ DAB = C ; Constr.

∴ ∠ C = the ∠ in the segt AHB. Ax. 1.

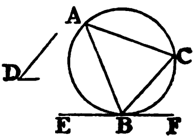
∴ *on the given straight line* AB *is described the segment* AHB *of a circle, which contains an angle equal to the given angle* C. [Q. E. F.]

PROP. XXXIV. PROB.

From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given ⊙, and D the given ∠: it is req[d] to cut off from the ⊙ ABC a seg[t] that shall contain an ∠ = the given ∠ D.

17.3. Draw the | EF touching the ⊙ ABC in the p[t] B, and at the p[t] B, in | BF,
23.1. make ∠ FBC = ∠ D: the seg[t] BAC shall contain an ∠ = ∠ D.



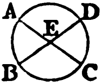
For, ∵ | EF touches ⊙ ABC,
and BC is drawn from the p[t] of contact B,
32.3. ∴ ∠ FBC = the ∠ in the alt. seg[t] BAC:
Constr. but ∠ FBC = ∠ D;
Ax. 1. ∴ ∠ D = the ∠ in the alt. seg[t] BAC.

∴ *from the given circle* ABC, *is cut off the segment* BAC, *containing an angle equal the given angle* D. [Q.E.F.]

PROP. XXXV. THEOR.

If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two |^s AC, BD cut each other in the p^t E, within the ⊙ ABCD; then shall the rect. AE. EC = the rect. BE. ED.

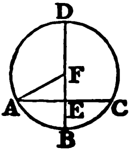


If AC, BD each pass through the cent., so that E is that cent.; then, Def. 15 1.

$$AE = EC = BE = ED,$$

and ∴ AE. EC. = BE. ED.

But let one of them BD pass through the cent., and cut the other AC, w^h does not pass through the cent., at r^t ∠^{s}, in the p^t E: if BD be bis^d in F, F is the cent.: join AF: then,



∵ BD passes through the cent., and cuts AC, w^h does not pass through the cent., at r^t ∠^{s}, in p^t E,

∴ AE = EC: 3. 3.

And,

∵ | BD is cut into two equal parts in p^t F, and into two unequal parts in p^t E,

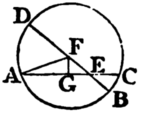
$$\therefore \quad BE.\ ED + EF^2 = FB^2 \qquad 5.\ 2.$$
$$= FA^2$$
$$= AE^2 + EF^2: \qquad 47.\ 1.$$

take away the com. part EF^2:

then the rem^g rect. BE. ED = AE^2, Ax. 3.

i. e. BE. ED = AE. EC.*

Next, let BD, w^h passes through the cent., cut the other AC, w^h does not pass through the cent., in E, but not at r^t ∠^{s}: then, as before, if BD be bis^d in F, F is the cent. of the ⊙.



* Because it has been proved that AE = EC.

12. 1. Join AF, and draw FG $\perp$ to AC: then,

3. 3. $AG = GC$;

5. 2. and $\therefore AE.EC + EG^2 = AG^2$:

add FG^2: then,

$$AE.EC + EG^2 + FG^2 = AG^2 + FG^2:$$

47. 1. but $EG^2 + FG^2 = EF^2$,

and $AG^2 + FG^2 = AF^2$:

Ax. 2. $\therefore AE.EC + EF^2 = AF^2$

Def. 15. 1. $= FB^2$

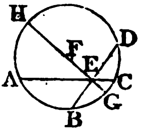
5. 2. $= BE.ED + EF^2$:

take away the com. part EF^2: then,

Ax. 3. $AE.EC = BE.ED$.

Lastly, let neither of the |s AC,

1. 3. BD, pass through the cent.; take the cent. F, and through E, the p^t of intersection of the |s AC, DB, draw the diam^r GEFH:



then, as has been shown above,

$AE.EC = GE.EH$.

and also $BE.ED = GE.EH$;

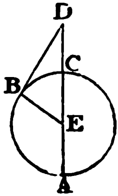
Ax. 1. $\therefore$ the rect. AE.EC. = the rect. BE.ED.

$\therefore$ *if two straight lines, &c.* [Q. E. D.]

PROP. XXXVI. THEOR.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

Let D be any p^t without the ⊙ ABC, and of the two |s DCA, DB, drawn from it, let DCA cut the ⊙, DB touch it: then shall the rect. AD. DC = DB^2.



Either DCA passes through the cent. of the ⊙, or it does not: first, let it pass through the cent. E, and join EB: then EBD is a $r^t\angle$: 18. 3.

and ∵ | AC is bis^d in p^t E,
and $prod^d$ to D,

∴ AD. DC + $EC^2 = ED^2$; 6. 2.

but ∵ EC = EB,

∴ $EC^2 = EB^2$;

and ∵ EBD is a $r^t\angle$,

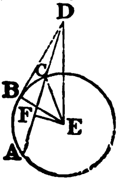
∴ $ED^2 = BD^2 + EB^2$: 47. 1.

∴ AD. DC + $EB^2 = BD^2 + EB^2$: Ax. 1.

take away the com. part EB^2;

then AD. DC = BD^2. Ax. 3.

But, if DCA do not pass through the cent. of the ⊙, take the cent. E, draw EF ⊥ to AC, and join EB, EC, ED: then 1. 3. 12. 1.



∵ | EF, w^h passes through the cent., cuts AC, w^h does not so pass, at $r^t\angle$ s,

∴ EF bis^{ts} AC in p^t F; 3. 3.

and ∵ | AC is bis^d in p^t F,
and $prod^d$ to D,

∴ AD. DC + $FC^2 = FD^2$: 6. 2.

add EF^2: then,

AD. DC + $FC^2 + EF^2 = FD^2 + EF^2$: Ax. 2.

but ∵ EFD is a $r^t\angle$,

∴ $ED^2 = FD^2 + EF^2$, 47. 1.

and also $EC^2 = FC^2 + EF^2$;

Ax. 1. and $\therefore$ $AD.DC + EC^2 = ED^2$:

but $\because$ $EC = ED$,

$\therefore$ $EC^2 = EB^2$,

also, $\because$ EBD is a $r^t \angle$,

47. 1. $\therefore$ $ED^2 = BD^2 + EB^2$;

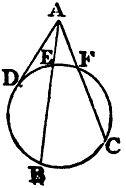
and $\therefore$ $AD.DC + EB^2 = BD^2 + EB^2$:

take away the com. part EB^2:

Ax. 3. then $AD.DC = BD^2$.

$\therefore$ *if from any point, &c.* [Q. E. D.]

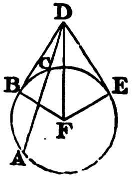
Cor.—If from any p[t] without a ⊙, there be drawn two |[s] cutting it, as AB, AC, the rect. contained by the whole |[s] and the parts of them without the ⊙[s] are = one another, viz. the rect. BA. AE = the rect. CA. AF: for each of them is = the sq. of the | AD, w[h] touches the ⊙.



PROP. XXXVII. THEOR.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.

Let D be any p[t] without the ⊙ ABC, and from it let two |[s] DCA, DB be drawn, of w[h] DCA cuts the ⊙, and DB meets it:
if the rect. AD. DC $= DB^2$,
DB shall touch the ⊙ ABC.



Find the cent. F of the ⊙ 1. 3.
draw the | DE touching it, 17. 3.
and join FE, FB, FD:
then, FED is a r[t] ∠: 18. 3.
and ∵ DE touches the ⊙,
and DCA cuts it,
∴ AD. DC $= DE^2$: 36. 3.
but AD. DC $= DB^2$; Hyp.
∴ $DE^2 = DB^2$, Ax. 1.
and DE = DB:

hence, in the △[s] DEF, DBF
{ side DE = DB, EF = BF,
and base FD is com. to both;
and ∴ ∠ DEF = ∠ DBF; 8. 1.
but ∠ DEF is a r[t] ∠;
∴ ∠ DBF is also a r[t] ∠; Ax. 1
and BF, if prod[d], is a diam[r];
but the | w[h] is drawn at r[t] ∠[s] to a diam[r],
from its extr[y], touches the ⊙: 16. 3.
∴ DB touches the ⊙ ABC.

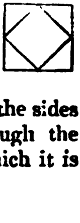
∴ *if from a point, &c.* [Q.E.D.]

BOOK IV.

DEFINITIONS.

I.

A RECTILINEAL figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

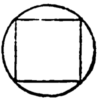


II.

In like manner a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

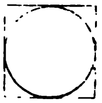
III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.



IV.

A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

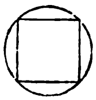


V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



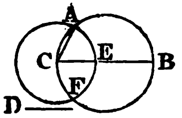
VII.

A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

In a given circle to place a straight line, equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given ⊙, and D the given |, w^h is not > the diam^r of the ⊙; it is req^d to place in ⊙ ABC a | = D.



Draw* a diam^r BC of the ⊙ ABC: then, if BC = D, the thing req^d is done; for in ⊙ ABC is placed a | BC = D.

Hyp. But, if not, BC is > D:

3. 1. make CE = D and from the cent. C, at the dist. CE, desc. the ⊙ AEF, and join CA: CA shall be = D.

For, ∵ C is the cent. of ⊙ AEF

Def. 15. ∴ CA = CE:

1. Constr but CE = D;

Ax. 1. ∴ CA = D.

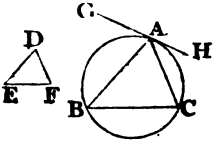
∴ *in the circle* ABC *is placed a straight line* CA *equal the given straight line* D. [Q. E. F.]

* Find the cent. and through it draw any | BC terminated both ways by the ⊙ce, this | will be a diamr.

PROP. II. PROB.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given $\odot$, DEF the given $\triangle$: it is req[d] to insc. in ABC a $\triangle$ equiang. to DEF.



Draw the | GAH touching the $\odot$ in the p[t] A; 1
at p[t] A, in the |[s] AG, AH, make $\angle$ GAB = $\angle$ DFE, 2
and $\angle$ HAC = $\angle$ DEF; and join BC: ABC shall be the $\triangle$ req[d].

For, $\because$ HAG touches the $\odot$ ABC,
and AC is drawn from the p[t] of contact,
$\therefore$ $\angle$ HAC = $\angle$ ABC in the alt. seg[t]: 32
but $\angle$ HAC = $\angle$ DEF; C
and $\therefore$ $\angle$ ABC = $\angle$ DEF; A
for the like reason,
$\angle$ ACB = $\angle$ DFE:
and $\therefore$ the rem[g] $\angle$ BAC = the rem[g] $\angle$ EDF. 32

$\therefore$ *the triangle* ABC *is equiangular to triangle* DEF, *and it is inscribed in circle* ABC. [Q. E. F.]

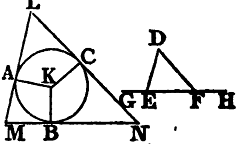
PROP. III. PROB.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given $\odot$, DEF the given $\triangle$:

it is req^{d} to desc. about $\odot$ ABC a $\triangle$ equiang. to $\triangle$ DEF.

Prod. EF both ways to the p^{ts} G, H ; find the
1. 3. cent. K of the $\odot$ ABC and from it draw any | KB;
at p^{t} K, in | KB, make
23. 1. $\angle$ BKA = $\angle$ DEG,
$\angle$ BKC = $\angle$ DFH ;
and through the p^{ts} A,
B, C, draw the |^{s} LAM,
17. 3. MBN, NCL, touching
the $\odot$ ABC : LMN shall be the $\triangle$ req^{d}.



For,

$\because$ LM, MN, NL touch $\odot$ ABC in the p^{ts} A, B, C,
to w^{h} from the cent. are drawn the |^{s} KA, KB, KC,
18. 3. $\therefore$ the $\angle^{s}$ at the p^{ts} A, B, C are r^{t} $\angle^{s}$:
and,

$\because$ the quadrilat^{l} fig. AMBK can, by drawing the | MK, be div^{d} into two $\triangle^{s}$,
$\therefore$ its four $\angle^{s}$ = four r^{t} $\angle^{s}$;
and of these four $\angle^{s}$, the two KAM, KBM are r^{t} $\angle^{s}$:
Ax. 1. $\therefore$ the other two $\angle^{s}$, (AKB + AMB) = two r^{t} $\angle^{s}$:
13. 1. but $\angle^{s}$ (DEG + DEF) = two r^{t} $\angle^{s}$;
Ax. 3. $\therefore$ $\angle^{s}$ (AKB + AMB) = $\angle^{s}$ (DEG + DEF);
Constr. but $\angle$ AKB = $\angle$ DEG :
Ax. 3. $\therefore$ the rem^{g} $\angle$ AMB = $\angle$ DEF.

In like manner it may be shown that
$\angle$ LNM = $\angle$ DFE ;
32. 1. and $\therefore$ the rem^{g} $\angle$ MLN = $\angle$ EDF.

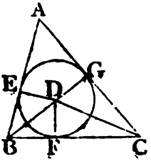
$\therefore$ *the triangle* LMN *is equiangular to triangle* DEF, *and it is described about circle* ABC.

[Q. E. F.]

PROP. IV. PROB.

To inscribe a circle in a given triangle.

Let △ ABC be given: it is req^d to insc. a ⊙ in it.



Bis^t the ∠^{s} ABC, ACB by the |^s BD, CD meeting one another in the p^t D, from w^h draw DE, DF, DG ⊥^{s} to AB BC, CA. 9. 1. 12. 1.

Then, in △^{s} EBD, FBD,

∵ { ∠ EBD = ∠ FBD, Constr.
r^t∠ DEB = r^t∠ DFB,
and side DB is opp. to ∠ DEB, in one △,
and to ∠ DFB in the other △;

∴ the other sides of the △^{s} are = one another; 26. 1.

and ∴ DE = DF:

for the like reason,

DG = DF;

and ∴ DG = DE;

thus, DE = DF = DG;

and ∴ the ⊙ desc^d from the cent. D, at the dist. of any one of these three |^s, will pass through the extr^s of the other two:

also, ∵ each of the ∠^{s} at the p^{ts} E, F, G, is a r^t∠,

∴ each of the |^s AB, BC, CA is drawn from the extr^y of a diam^r at r^t∠^{s} to it;

and ∴ each of these |^s touches the ⊙ EFG. 16. 3.

∴ *this circle* EFG *is inscribed in triangle* ABC.

[Q.E.F.]

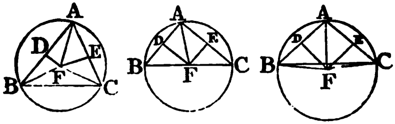
PROP. V. PROB.

To describe a circle about a given triangle.

Let ABC be the given △: it is req[d] to desc. a ⊙ about it.

10. 1. Bis[t] AB, AC in the p[ts] D, E, and from these
11. 1. p[ts] draw DF, EF at r[t] ∠[s] to AB, AC:

DF, EF prod[d] meet one another;
for, if they do not meet, they are ||;
and ∴ AB, AC, w[h] are at r[t] ∠[s] to them, are also ||,
w[h] is absurd:



let DF, EF meet in F, and join FA; also, if the p[t] F be not in BC, join BF, CF: then,

Constr. ∵ side AD = DB,
and DF is com. to △[s] ADF, BDF,
and also at r[t] ∠[s] to AB;
4. 1. ∴ the base AF = BF:
for like reasons, AF = CF;
and ∴ BF = CF.
Thus, AF = BF = CF.

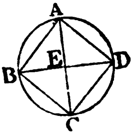
And ∴ *the circle described from the centre* F, *at the distance of any one of these straight lines, will pass through the extremities of the other two, and be described about the triangle* ABC. [Q. E. F.]

COR.—If the cent. of the ⊙ fall within the △, it is manifest that each of the ∠s of the △ is in a $seg^t >$ a $\frac{1}{2}$ ⊙; 1. a.
and ∴ each of these ∠s is < a r^t ∠:
but if the cent. be in one of the sides of the △, the ∠ opp. to this side is in a $\frac{1}{2}$ ⊙,
and ∴ this ∠ is a r^t ∠:
and, if the cent. falls without the △, the ∠ opp. to the side beyond w^h it is, is in a $seg^t <$ a $\frac{1}{2}$ ⊙,
and ∴ is > a r^t ∠:
hence, conversely, if the given △ be acute $∠^d$, the cent. of the ⊙ falls within it; if it be a r^t $∠^d$ △, the cent. is in the side opp. to the r^t ∠; and if it be an obt. $∠^d$ △, the cent. falls without the △, beyond the side opp. to the obt. ∠.

PROP. VI. PROB.

To inscribe a square in a given circle.

Let ABCD be the given ⊙: it is req^d to insc. a sq. in it.



E being the cent. of the ⊙, draw the $diam^{rs}$ AC, BD at r^t ∠s to each other, and join AB, BC, CD, DA: the fig. ABCD shall be the sq. req^d.

For, ∵ side BE = ED, and EA is com. to △s EAB, EAD, and at r^t ∠s to BD:
∴ the base BA = AD:

for like reasons,

BC, CD are each = BA or AD ;

and ∴ the quadrilat[l] fig. ABCD is equilat[l]

Also, ∵ BD is a diam[r] of the ⊙,

∴ BAD is a $\frac{1}{2}$ ⊙,

31. 3. and ∴ ∠ BAD is a r[t] ∠:

for the same reason,

each of the ∠[s] ABC, BCD, CDA is a r[t] ∠:

∴ the fig. ABCD is rectangular :

and it has been shown to be equilat[l].

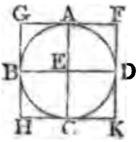
Def. 30. ∴ *it is a square, and it is inscribed in circle*

1. ABCD. [Q. E. F.]

PROP. VII. PROB.

To describe a square about a given circle.

Let ABCD be the given ⊙: it is req[d] to desc. a sq. about it.



E being the cent. of the ⊙, draw two diam[rs] AC, BD at r[t] ∠[s] to each other, and through the p[ts] A, B, C, D, draw FG, GH,

17. 3. HK, KF, touching the ⊙: the fig. GHKF shall be the sq. req[d].

For, ∵ FG touches the ⊙,

and EA is drawn from the cent. to the p[t] of contact

18. 3. ∴ the ∠[s] at A are r[t] ∠[s]:

for the same reason,

the $\angle$s at B, C, D are r^t $\angle$s:

And $\because$ AEB is a r^t $\angle$, as is also EBG,

$\therefore$ GH is $\parallel$ AC; 28. 1.

for the same reason, FK is $\parallel$ AC:

and in like manner it may be shown that

GF, HK are each $\parallel$ BED:

$\therefore$ the figs. GK, GC, AK, FB, BK are $\square$s;

and $\therefore$ GF = HK, GH = FK; 34. 1.

and $\because$ AC = BD,

and that AC = each of the two GH, FK,

BD = each of the two GF, HK;

$\therefore$ GH = FK = GF = HK,

and $\therefore$ the quadrilat^l fig. FGHK is equilat^l.

Again,

$\because$ GBEA is a $\square$, and AEB is a r^t $\angle$,

$\therefore$ AGB is also a r^t $\angle$, 34. 1.

and in the same manner it may be shown that

the $\angle$s at H, K, F are r^t $\angle$s:

$\therefore$ the fig. FGHK is rectangular:

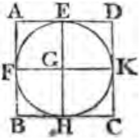
and it has been shown to be equilat^l.

$\therefore$ *it is a square; and it is described about the circle* ABCD. [Q. E. F.]

PROP. VIII. PROB.

To inscribe a circle in a given square.

Let ABCD be the given sq.: it is req^d to insc. a $\odot$ in it.



10. 1. Bis^t the sides AB, AD in the p^{ts} F, E; through E draw EH
11. 1. $\parallel$ AB or DC, and through F draw FK $\parallel$ AD or BC: then each of the fig^s AK, KB, AH, HD, AG, GC, BG, GD, is a $\square$; and
34. 1. $\therefore$ any side $=$ that opp. to it:
Def. 30. 1. And $\because$ AD $=$ AB,
and that AE $= \frac{1}{2}$AD
AF $= \frac{1}{2}$AB,
Ax. 7. $\therefore$ AE $=$ AF:
but FG $=$ the opp. side AE;
GE $=$ the opp. side AF,
and $\therefore$ FG $=$ GE:
in the same manner it may be shown that
GH, GK each $=$ GF or GE:
$\therefore$ GE $=$ GF $=$ GH $=$ GK;
and $\therefore$ the $\odot$ desc^d from the cent. G, at the dist. of any one of these four $|^s$, will pass through the extr^s of the other three:

And
29. 1. $\because$ the $\angle^s$ at the p^{ts} E, F, H, K are r^t $\angle^s$,
Cor. 16. 3 and that the $|$ drawn from the extr^y of a diam^r at r^t $\angle$ to it touches the $\odot$,
$\therefore$ each of the $|^s$ AB, BC, CD, DA touches the $\odot$.

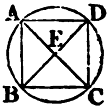
And $\therefore$ *the circle is inscribed in the square* ABCD. [Q. E. F.]

PROP. IX. PROB.

To describe a circle about a given square.

Let ABCD be the given sq.: it is req[d] to desc a ⊙ about it.

Join AC, BD, cutting one another in E: then in the two △s DAC, BAC,



$\because \begin{cases} \text{side AD} = \text{AB}, \\ \text{side AC is com.} \\ \text{and base DC} = \text{BC}; \end{cases}$

$\therefore$ ∠ DAC = ∠ BAC, 8. 1.

i. e. ∠ DAB is bis[d] by | AC:

in like manner it may be shown that each of the ∠s ABC, BCD, CDA is bis[d] by the |s BD, AC:

Hence, $\because$ ∠ DAB = ∠ ABC, Def. 30 I.

and that ∠ EAB = ½ ∠ DAB,

∠ EBA = ½ ∠ ABC;

$\therefore$ ∠ EAB = ∠ EBA; Ax. 7

$\therefore$ side EA = side EB: 6. 1.

and in like manner it may be shown that

EC, ED each = EA or EB:

$\therefore$ EA = EB = EC = ED.

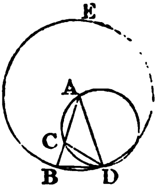
And $\therefore$ *the circle described from the centre* E, *at the distance of any one of these four straight lines, will pass through the extremities of the other three, and be described about the square* ABCD.

[Q. E. F.]

PROP. X. PROB.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

11. 2. Take any | AB, and div. it in p^t C, so that the rect. AB. BC $= AC^2$; from cent. A, at dist. AB, desc. the
1. 4. ⊙ BDE: in it place the | BD $=$ AC, w^h is not $>$ the diam^r of the ⊙, and join DA: the △ ABD shall be such as is req^d, *i. e.* each of the $\angle^s$ ABD ADB shall be double of the third $\angle$ BAD.



5. 4. Join DC, and about the △ ADC desc. the ⊙ ACD:

Constr. then, ∵ AB. BC $= AC^2$,
and that AC $=$ BD,
∴ AB. BC $= BD^2$:

and, ∵ from the p^t B, without the ⊙ ACD, two |^s BCA, BD, are drawn to the ⊙ ^{ce}, of w^h BCA cuts the ⊙ in C, and BD meets it in D,
and that the rect. AB. BC $= BD^2$,

37. 3. ∴ BD touches the ⊙ ACD:
and ∵ BD touches the ⊙,
and DC is drawn from the p^t of contact D,

32. 3. ∴ $\angle$ BDC $= \angle$ DAC in the alt. seg^t:
add $\angle$ CDA:
then the whole $\angle$ BDA $= \angle^s$(CDA $+$ DAC)

32. 1. $=$ the ext^r $\angle$ BCD;

but $\because$ side $AD = AB$,

$\therefore$ $\angle BDA = \angle CBD$; 5. 1

$\therefore$ $\angle CBD = \angle BCD$,

and $\therefore \angle BDA = \angle DBA = \angle BCD$:

and $\because$ $\angle CBD = \angle BCD$,

$\therefore$ side $DB =$ side DC: 6. 1.

but $DB = CA$;

$\therefore$ also $CA = CD$,

and $\therefore$ $\angle CDA = \angle CAD$; 5. 1

$\therefore \angle^s(CDA + DAC) = 2 \angle DAC$:

but $\angle^s(CDA + DAC) = \angle BCD$; 32. 1.

$\therefore$ also $\angle BCD = 2 \angle DAC$:

and $BCD =$ each of the $\angle^s$ ADB, ABD;

$\therefore$ each of the $\angle^s$ $ADB, ABD = 2 \angle DAB$.

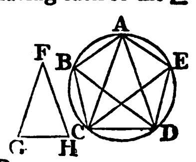
$\therefore$ an isosceles triangle ABD *is described, having each of the angles at the base double of the third angle.* [Q. E. F.]

PROP. XI. PROB.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given $\odot$: it is req[d] to insc. an equilat[l] and equiang[r] pntg in it.

Desc. an isosc. $\triangle$ FGH, having each of the $\angle^s$ at G, H double of that at F; 10. 4.
and in $\odot$ ABCDE insc. the $\triangle$ACD equiang[r] to FGH, so that $\angle$ CAD = the $\angle$ at F, and the $\angle^s$ ACD, CDA be each = that at G or H, and $\therefore$ be each double of CAD. 2. 4.



9. 1. Bis[t] the $\angle^s$ ACD, ADC by the $|^s$ CE, DB ; and join AB, BC, DE, EA: ABCDE shall be the pntg[n] req[d].

For,

$\because$ each of the $\angle^s$ ACD, ADC is double of CAD, and that they are bis[a] by the $|^s$ CE, DB;

$\therefore \angle$ DAC = ACE = ECD = CDB = BDA :

26. 3. but equal $\angle^s$ stand on equal arcs ;

$\therefore$ arc AB = BC = CD = DE = EA :

29. 3. and equal arcs are subtended by equal $|^s$;

$\therefore$ | AB = BC = CD = DE = EA :

$\therefore$ the pntg[n] ABCDE is equilat[l].

Also, from above, arc AB = DE;

add arc BCD;

then, the whole arc ABCD = the whole EDCB

but the $\angle$ AED stands on arc ABCD,

and the $\angle$ BAE stands on arc EDCB ;

27. 3. $\therefore \angle$ AED = $\angle$ BAE:

for the same reason, each of the three $\angle^s$ ABC, BCD, CDE = $\angle$ BAE or AED:

$\therefore$ the pntg ABCDE is equiang[r] ;

and it has been shown to be equilat[l].

$\therefore$ *in the given circle an equilateral and equiangular pentagon has been inscribed.*

[Q.E.F.]

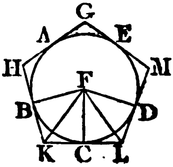
PROP. XII. PROB

To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given ⊙: it is req[d] to desc. an equilat[l] and equiang[r] pntg[l] about it.

Let the ∠[s] of an equilat and equiang[r] pntg[n], insc[d] in the ⊙, be in the p[ts] A, B, C, D, E, so that the arc $AB = BC = CD = DE = EA$; and through the p[ts] draw the |[s] GH, HK, KL, LM, MG, touching the ⊙: the fig. GHKLM shall be the pntg[n] req[d]. 11.4. 17.3.

Take the cent. F, and join FB, FK, FC, FL, FD
then ∵ KL touches the ⊙ in p[t] C,
and to C is drawn the | FC from the cent. F;
∴ FC is ⊥ to KL, 18.3.
and ∴ the ∠[s] at C are r[t] ∠[s],
and ∵ FCK, FBK are r[t] ∠[s],



∴ $FK^2 = FC^2 + KC^2$ 47.1.
$FK^2 = FB^2 + BK^2$
∴ $FC^2 + KC^2 = FB^2 + BK^2$
but $FC^2 = FB^2$
∴ rem[r] KC^2 = rem[r] BK^2,
and $KC = BK$:
hence, in the two △[s] FKC, FKB,
∵ { side $FC = FB$,
FK is com. to both,
base KC = base BK;
∴ ∠ CKF = BKF, ∠ CFK = BFK; 8.1.
and ∴ ∠ CKB is double of CKF,
∠ CFB double of CFK:
for the same reason,
∠ CFD is double of CFL, ∠ CLD double of CLF;
but ∵ the arc BC = CD,
∴ ∠ BFC = CFD; 27.3.

and, from above, BFC is double of KFC,
CFD double of CFL;

Ax. 7. $\therefore\ \angle$ KFC = CFL:
also rt $\angle$ FCK = rt $\angle$ FCL:
hence, in the two $\triangle^s$ FKC, FLC,

$\because$ two $\angle^s$ of the one = two $\angle^s$ of the other, each to each,
and the side FC, w^h is adj^t to the equal $\angle^s$ in each $\triangle$, is com. to both;

26. 1. $\therefore$ the third $\angle$ FKC = the third $\angle$ FLC;
and the other sides = the other sides;
$\therefore$ KC = CL,
and $\therefore$ KL is double of KC:
in the same manner it may be shown that
HK is double of BK.

And $\because$ KB = KC, as is shown above,
and that KL is double of KC, HK double of BK,

Ax. 6. $\therefore$ HK = KL:
in like manner it may be shown that
GH, GM, ML each = HK or KL:
and $\therefore$ the $pntg^n$ GHKLM is $equilat^l$.

Also, $\because$ $\angle$ FKC = $\angle$ FLC,
and that $\angle$ HKL is double of FKC,
$\angle$ KLM double of FLC,

Ax. 6. $\therefore$ $\angle$ HKL = $\angle$ KLM:
and in like manner, it may be shown that
the $\angle^s$ KHG, HGM, GML each = HKL or KLM:
$\therefore\ \angle$ GHK = HKL = KLM = LMG = MGH.

And $\therefore$ *the pentagon* GHKLM *is equiangular, and it has been shown to be equilateral; and it is described about the circle* ABCDE.

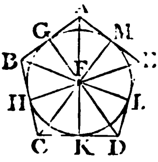
[Q. E. F.]

PROP. XIII. PROB.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilat^{l} and equiang^{r} pntg^{n}: it is req^{d} to insc. a $\odot$ in it.

Bis^{t} the $\angle^{s}$ BCD, CDE by the $|^{s}$ CF, DF, 9. 1.
and from the p^{t} F, in w^{h} they meet, draw the $|^{s}$ FB, FA, FE; then,



in the two $\triangle^{s}$ BCF, DCF,

$\because \begin{cases} \text{side BC} = \text{CD}, \\ \text{side CF is com.} \\ \text{and } \angle\ \text{BCF} = \angle\ \text{DCF}: \end{cases}$ Hyp.

$\therefore \begin{cases} \text{the base BF} = \text{base FD}, \\ \text{and } \angle\ \text{CBF} = \angle\ \text{CDF}: \end{cases}$ 4. 1.

and $\because$ $\angle$ CDE is double of CDF, Constr
and that CDE = CBA, and CDF = CBF;
$\therefore$ $\angle$ CBA is also double of CBF;
and $\therefore$ $\angle$ CBA is bis^{d} by $|$ BF:
in like maner it may be shown that
the $\angle^{s}$ BAE, AED, are bis^{d} by the $|^{s}$ AF, EF.
From the p^{t} F, draw FG, FH, FK, FL, FM. 12. 1.
$\perp^{s}$ to the $|^{s}$ AB, BC, CD, DE, EA:
then, in the two $\triangle^{s}$ FHC, FKC,

$\because \begin{cases} \angle\ \text{HCF} = \text{KCF, r}^{t} \angle\ \text{FHC} = \text{r}^{t} \angle\ \text{FKC}, \\ \text{and also the side FC, w}^{h} \text{ is opp. to one of the} \\ \text{equal } \angle^{s} \text{ in each } \triangle, \text{ is com. to each } \triangle, \end{cases}$

$\therefore$ the other sides = the other sides, each to each, 26. 1.
and $\therefore$ the $\perp$ FH = the $\perp$ FK:
in the same manner it may be shown that
FL, FM, FG each = FH or FK;
$\therefore$ FG = FH = FK = FL = FM:

and $\therefore$ the $\odot$ desc[d] from the cent. F, at the dist. of any one of these five |[s] will pass through the extr[s] of the other four:

and $\because$ the $\angle$[s] at the p[ts] G, H, K, L, M, are r[t] $\angle$[s],
$\therefore$ each of these |[s] is drawn from the extr[y] of a diam[r] of the $\odot$ at r[t] $\angle$[s] to that diam[r];

3. and $\therefore$ each of the |[s] touches the $\odot$.

$\therefore$ *the circle is inscribed in the pentagon* ABCDE.
[Q. E. F.]

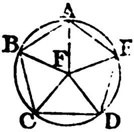
PROP. XIV. PROB.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilat[l] and equiang[l] pntg[n]: it is is req[d] to desc. a $\odot$ about it.

9. 1. Bis[t] the $\angle$[s] BCD, CDE by the |[s] CF, DF, and from the p[t] F, in w[h] they meet, draw the |[s] FB, FA, FE.

It may be shown, as in the last prop[n], that the $\angle$[s] CBA, BAE, AED are bis[d] by the |[s] FB, FA, FE:



and $\because$ $\angle$ BCD $=$ $\angle$ CDE,
and that $\angle$ FCD $= \frac{1}{2}$ $\angle$ BCD,
$\angle$ CDF $=$ $\angle$ CDE:

∴x.7. $\therefore$ $\angle$ FCD $=$ $\angle$ CDF;

6. 1. and $\therefore$ side FC $=$ side FD:

in like manner it may be shown that
FB, FA, FE each $=$ FC or FD;
$\therefore$ FA $=$ FB $=$ FC $=$ FD $=$ FE.

And $\therefore$ *the circle described from the centre* F, *at the distance of any one of these five straight lines, will pass through the extremities of the other four, and be described about the pentagon* ABCDE.

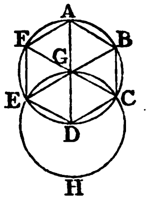
[Q.E.F.]

PROP. XV. PROB.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given ⊙: it is req[d] to insc. an equilat[l] and equiang[r] hxg[n] in it.

Find the cent. G of the ⊙ ABCDEF, and draw the diam[r] AGD; from the cent. D, at the dist. DG, desc. the ⊙ EGCH; join EG, CG, prod. them to the p[ts] B, F; and join AB, BC, CD, DE, EF, FA: the hxg[n] ABCDEF shall be equilat[l] and equiang[r]. 1. 3.



For,

∵ G is the cent. of ⊙ ABCDEF,

$\therefore$ GE = GD;

and,

∵ D is the cent. of ⊙ EGCH,

$\therefore$ DE = GD:

$\therefore$ GE = DE,

and the △ EGD is equilat[l];

and $\therefore$ its three ∠[s] are equal to one another: Cor. 5.

but the three ∠[s] of a △ = two r[t] ∠[s]; 32. 1.

and $\therefore$ the ∠ EGD is the third part of two r[t] ∠[s]:

in the same manner it may be shown that

the ∠ DGC is also the third part of two r[t] ∠[s]:

13. 1. and ∵ the adj^t ∠^{s} (EGC + CGB) = two r^t ∠^{s};
∴ the rem^g ∠ CGB is the third part of two r^t ∠^{s}:
∴ ∠ EGD = DGC = CGB:
15. 1. and these ∠^{s} = their opp. ∠^{s} BGA, AGF, FGE:
∴ ∠ EGD = DGC = CGB = BGA = AGF = FGE:
26. 3. but equal ∠^{s} stand on equal arcs;
∴ arc AB = BC = CD = DE = EF = FA:
29. 3. and equal arcs are subtended by equal |^s;
∴ the six sides of the hxg ABCDEF are equal to one another, and the hxg is equilat^l.

It is also equiang^r: for,
to the equal arcs AF, ED add the arc ABCD;
then the whole arc FABCD = the whole EDCBA:
and the ∠ FED stands on the arc FABCD,
and the ∠ AFE on EDCBA,
27. 3. ∴ ∠ FED = ∠ AFE:
in the same manner it may be shown that
the other ∠^{s} of the hxg^n each = ∠ AFE or FED.

And ∴ *the hexagon is equiangular; and it has been shown to be equilateral; and it is inscribed in the given circle* ABCDEF. [Q. E. F.]

Cor.—From this it is manifest that the side of the hxg^n is equal to the | from the cent. *i. e.* to the semi-diam^r of the ⊙.

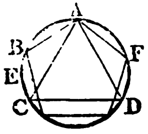
And if through the p^{ts} A, B, C, D, E, F there be drawn |^s touching the ⊙, an equilat and equiang^r hxg^n will be desc^d about it, w^h may be dem^d from what has been said of the pntg: and likewise a ⊙ may be insc^d in a given equilat^l and equiang^r hxg^n, and desc^d about it, by a method like to that used for the pntg^n.

PROP. XVI. PROB.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given ⊙: it is req^{d} to insc. an equilat^{l} and equiang^{r} quindecg^{n} in it.

Let AC be the side of an equilat^{l} △ insc^{d} in the ⊙; AB the side of an equilat^{l} and equiang^{r} pntg insc^{d} in the same: then, 2. 4 11. 4.



∵ the arc ABC is the third part of the whole ⊙^{ce},

∴ of such equal parts as the whole ⊙^{ce} contains fifteen, the arc ABC contains five;

and the arc AB, w^{h} is the fifth part of the whole, contains three such parts;

∴ the difference BC contains two of these parts:

bis^{t} BC in E: 30. 3.

then BE, EC are, each of them, the fifteenth part of the whole ⊙^{ce} ABCD,

∴ *if the straight lines* BE, EC *be drawn, and straight lines equal to them be placed round in the* 1. 4. *whole circle, an equilateral and equiangular quindecagon will be inscribed in the circle.*

[Q. E. F.]

And in the same manner as was done in the pntg^{n}, if through the p^{ts} of division made by insc^{g} the quindcg^{n}, |^{s} be drawn touching the ⊙, an equilat^{l} and equiang^{r} quindcg^{n} will be desc^{d} about it: and likewise, as in the pntg^{n}, a ⊙ may be insc^{d} in a given equilat^{l} and equiang^{r} quindcg^{n}, and desc^{d} about it.

BOOK V.

DEFINITIONS

I.

A LESS magnitude is said to be a part of a greater magnitude, when the less measures the greater. that is, 'when the less is contained a certain 'number of times exactly in the greater.'

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, when the greater contains the less a 'certain number of times exactly.'

III.

'Ratio is the mutual relation of two magnitudes of 'the same kind to one another, in respect of 'quantity.'

IV.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever

of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth: or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth: or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

VI.

Magnitudes which have the same ratio are called proportionals. 'N. B. When four magnitudes 'are proportionals, it is usually expressed by 'saying, the first is to the second as the third to 'the fourth.'

VII.

When, of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

VIII.

Analogy or proportion, is the similitude of 'ratios.'

IX.

Proportion consists in three terms at least.

X.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c., increasing the denomination still by unity, in any number of proportionals.

Definition A, to wit of compound ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C the same ratio that G has to H; and C to D the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the

ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

'Geometers make use of the following technical 'words, to signify certain ways of changing either 'the order or magnitude of proportionals, so that 'they continue still to be proportionals.'

XIII.

Permutando, or alternando, by permutation or alternately. This word is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shown in the 16th Prop. of this fifth Book.

XIV.

Invertendo, by inversion; when there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third. Prop. B. Book 5.

XV.

Componendo, by composition; when there are four proportionals, and it is inferred, that the first together with the second, is to the second, as the third together with the fourth, is to the fourth. 18th Prop. Book 5.

XVI.

Dividendo, by division; when there are four proportionals, and it is inferred, that the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth. 17th Prop. Book 5.

XVII.

Convertendo, by conversion; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E. Book 5.

XVIII.

Ex æquali (sc. distantiâ), or ex æquo, from equality of distance; when there is any number of magnitudes more than two, and as many other, such that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: 'Of this 'there are the two following kinds, which arise 'from the different order in which the magnitudes 'are taken, two and two.'

XIX.

Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to

the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in the 22nd Prop. Book 5.

XX.

Ex æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion.* This term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order: and the inference is as in the 18th definition. It is demonstrated in the 23rd Prop. of Book 5.

AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes, of which the same or equal

* Prop. lib. 2. Archimedis de sphærâ et cylindro.

magnitudes are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the others.

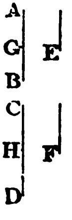
Let any n° of magn^s AB, CD be equimult^s of as many others, E, F, each of each: whatsoever mult. AB is of E, the same mult. shall $AB+CD$ be of $E+F$.

For,

$\because$ AB is the same mult. of E that CD is of F,

$\therefore$ as many magn^s as there are in AB, each $=$ E,
so many are there in CD, each $=$ F.

Div. AB into magn^s AG, GB, each $=$ E, and CD into magn^s CH, HD, each $=$ F: then, the n° of the magn^s CH, HD, shall be $=$ the n° of the others AG, GB:



And $\because$ $AG = E$, and $CH = F$,

Ax. 2. 1. $\therefore$ $AG + CH = E + F$:

also, $\because$ $GB = E$, and $HD = F$,

$\therefore$ $GB + HD = E + F$:

$\therefore$ as many magn^s as there are in AB,
each $=$ E,

so many are there in AB + CD, each = E + F:
∴ whatsoever mult. AB is of E,
the same mult. is AB + CD of E + F.

∴ *if any magnitudes, how many soever, be equimultiples of as many, each of each, whatsoever multiple any one of them is of its part, the same shall all the first magnitudes be of all the others.*

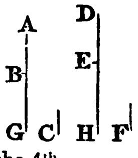
For the same demonstration holds in any nº of magn^s^ w^h^ has here been applied to two.

[Q. E. D.]

PROP. II. THEOR.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let AB the 1st be the same mult. of C the 2nd, that DE the 3rd is of F the 4th; and BG the 5th be the same mult. of C the 2nd, that EH the 6th is of F the 4th: then shall AG (the 1st + the 5th) be the same mult. of C the 2nd, that DH (the 3rd + the 6th) is of F the 4th.



For,

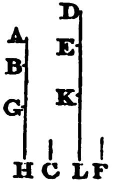
∵ AB is the same mult. of C that DE is of F,
∴ there are as many magn^s^ in AB, each = C,
as there are in DE, each = F:

similarly, as many as there are in BG, each = C,
so many there are in EH, each = F ;

∴ as many magn^s as there are in the whole AG, each = C,
so many are there in the whole DH, each = F ;

∴ AG is the same mult. of C that DH is of F,
i. e. AG (the 1st + the 5th)
is the same mult. of C the 2nd,
that DH (the 3rd + the 6th), is of F the 4th.

∴ *if the first be the same multiple of the second, &c.* [Q. E. D.]



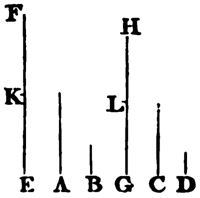
Cor.—From this it is plain, that if any n^o of magn^s AB, BG, GH, be mult^s of another C ; and as many DE, EK, KL be the same mult^s of F, each of each :
then, the whole of the 1st, viz. AH, is the same mult. of C, that the whole of the last, viz. DL, is of F.

PROP. III. THEOR.

If the first be the same multiple of the second, which the third is of the fourth ; and if of the first and third there be taken equimultiples ; these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the 1st be the same mult. of B the 2nd, that C the 3rd is of D the 4th ; and of A, C let

equimult^s EF, GH be taken: then shall EF be the same mult. of B that GH is of D.



For,

∵ EF is the same mult. of A, that GH is of C;

∴ there are in EF as many magn^s, each = A, as there are in GH, each = C:

let EF be div^d into the magn^s EK, KF, each = A; and GH into those GL, LH, each = C: then,

the n^o of the magn^s EK, KF $\Big\} = \Big\{$ that of the others GL, LH:

And

∵ A is the same mult. of B, that C is of D, and that EK = A, GL = C:

∴ EK is the same mult. of B, that GL is of D:

for the same reason,

KF is the same mult. of B, that LH is of D:

and so on, if in EF, GH there be more parts = A, C:

Hence,

∵ EK the 1^st is the same mult. of B the 2^nd, w^h GL the 3^rd is of D the 4^th,

and that KF the 5^th is the same mult. of B the 2^nd, w^h LH the 6^th is of D the 4^th;

∴ EF (the 1^st + the 5^th) is the same mult. of B the 2^nd, 2. 5.
w^h GH (the 3^rd + the 6th) is of D the 4^th.

∴ *if the first, &c.* [Q. E. D.]

PROP. IV. THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth, then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz., 'the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'

Let A the 1^{st} have to B the 2^{nd} the same r^o w^h C the 3^{rd} has to D the 4^{th}: and of A, C let there be taken any equimult^s whatever E, F; and of B, D any equimult^s whatever G, H: then E shall have the same r^o to G w^h F has to H.

Take of E, F any equimult^s whatever K, L; and of G, H any equimult^s whatever M, N: then,

∵ E is the same mult. of A, that F is of C; and of E, F have been taken equimult^s K, L;

3. 5. ∴ K is the same mult. of A, that L is of C: for the same reason,

M is the same mult. of B, that N is of D.

Hyp. And ∵ A : B :: C : D,

and of A, C have been taken certain equimult^s K, L of B, D have been taken certain equimult^s M, N;

$\therefore$ as K is $>, =$ or $<$ M,
so L is $>, =$ or $<$ N : Def.5.5.

but K, L are any equimult[s] whatever of E, F; Constr.
M, N any whatever of G, H:

and $\therefore$ E : G :: F : H. Def.5.5.

$\therefore$ *if the first, &c.* [Q. E. D.]

Cor.—Likewise, if the 1[st] has the same r[o] to the 2[nd], w[h] the 3[rd] has to the 4[th]; then also, any equimult[s] of the 1[st] and 3[rd] shall have the same r[o] to the 2[nd], and 4[th]; and in like manner, the 1[st] and the 3[rd] shall have the same r[o] to any equimult[s] of the 2[nd] and 4[th].

Let A the 1[st] have to B the 2[nd] the same r[o] w[h] the 3[rd] C has to the 4[th] D; and of A and C let E and F be any equimult[s] whatever: then shall
E : B :: F : D.

Take of E, F any equimult[s] whatever K, L;
and of B, D any equimult[s] whatever G, H:

then it may be dem[d], as before, that
K is the same mult. of A, that L is of C:

And $\because$ A : B :: C : D, Hyp.

and,
of A, C certain equimult[s] K, L have been taken,
and of B, D certain equimult[s] G, H;

$\therefore$ as K is $>, =$ or $<$ G,
so L is $>, =$ or $<$ H : Def.5.5.

but K, L are any equimult[s] whatever of E, F; Constr.
and G, H any whatever of B, D;

$\therefore$ E : B :: F : D. Def.5.5.

And in the same way the other case is dem[d].

PROP. V. THEOR.

If one magnitude be the same multiple of anoth which a magnitude taken from the first is of magnitude taken from the other; the remaind shall be the same multiple of the remainder, th the whole is of the whole.

Let the magn. AB be the same mult. of CD, th AE taken from the 1st is of CF taken from the other: the rem. EB shall be the same mult. of the rem. FD, that the whole AB is of the whole CD.

Take AG the same mult. of FD, that AE is of CF:

1. 3. ∴ AE is the same mult. of CF, that EG is of CD:

but, by the hyp.,

AE is the same mult. of CF, that AB is of (

∴ EG is the same mult. of CD, that AB is of (

Ax. 1.5. and ∴ EG = AB:

take from each the com. magn. AE:

then the rem. AG = the rem. EB.

Hence,

Constr. ∵ AE is the same mult. of CF that AG is of]

and that AG = EB;

∴ AE is the same mult. of CF, that EB is of :

Hyp. but AE is the same mult. of CF, that AB is of (

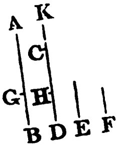
∴ EB is the same mult. of FD, that AB is of (

∴ *if one magnitude, &c.* [Q. E. D

PROP. VI. THEOR.

...o magnitudes be equimultiples of two others, ...nd if equimultiples of these be taken from the ...irst two; the remainders are either equal to ...these others, or are equimultiples of them.

Let the two magn^s AB, CD be equimult^s of the ...o E, F; and let AG, CH taken from the first two ...e equimult^s of the same E, F: ...e rem^{rs} GB, HD shall be either = E, F, or be equimult^s of them.



From the hyp., GB must be either = E, or a mult. of it.

First, let GB = E; then shall HD = F.

Make CK = F: then

∵ AG is the same mult. of E that CH is of F, and that GB = E, and CK = F;

∴ AB is the same mult. of E, that KH is of F: Hyp.

but AB is the same mult. of E, that CD is of F;

∴ KH is the same mult. of F, that CD is of F; Ax. 1.5

and ∴ KH = CD:

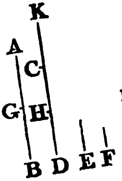
take away the com. magn. CH;

then the rem^r KC = the rem^r HD; Cons

but KC = F;

∴ HD = F.

Next, let GB be a mult. of E: HD shall be the same mult. of F.



Make CK the same mult. of F, that GB is of E: then, H

∵ AG is the same mult. of E, that CH is of F;

and GB the same mult. of E, that CK is of F;

∴ AB is the same mult. of E, that KH is of F:
but AB is the same mult. of E, that CD is of F;
∴ KH is the same mult. of F, that CD is of F;
1.5. and ∴ KH = CD:
take away CH from both;
then the remr KC = the remr HD:
and,
onstr. ∵ GB is the same mult. of E, that KC is of F;
and that KC = HD;
∴ HD is the same mult. of F, that GB is of E.

∴ *if two magnitudes, &c.* [Q. E. D.]

PROP. A. THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Take any equimults of each of the magns, as the doubles of each: then, by defn 5th of this Book.
if the double of the 1st be > the double of the 2
the double of the 3rd is > the double of the 4
but if the 1st be > the 2nd,
the double of the 1st is > the double of the
∴ also the double of the 3rd is > the double of the
and ∴ the 3rd is > the 4th:
In like manner,
if the 1st be = or < the 3rd,
it can be proved that
the 3rd is = or < the 4th.

∴ *if the first, &c.* [Q. E.

PROP. B. THEOR.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

Let $A : B :: C : D$; then also,
inv^{ly}, $B : A :: D : C$.

Take of B, D any equimult^s E, F;
and of A, C any equimult^s G, H;
and first let E be $>$ G,
i.e. G be $<$ E:

then, $\because A : B :: C : D$, Hyp.
and of A, C, the 1st and 3rd,
G, H are equimult^s,
and of B, D, the 2nd and 4th,
E, F are equimult^s;
and that G is $<$ E;
$\therefore$ H is $<$ F; Def.5
i.e. F is $>$ H;
$\therefore$ if E be $>$ G,
F is $>$ H:

in like manner,
if E be $=$ or $<$ G,
it may be shown that
F is $=$ or $<$ H:

but E, F are any equimult^s whatever of B, D, c
G, H any whatever of A, C;
and $\therefore B : A :: D : C$. I

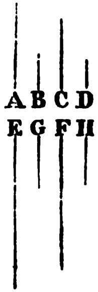
$\therefore$ *if four magnitudes, &c.* [Q. E. D.]

PROP. C. THEOR.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.

Let A the 1st be the same mult. of B the 2nd, that C the 3rd is of D the 4th: then,

$$A : B :: C : D.$$



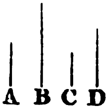
Take of A, C any equimult^s E, F;
and of B, D any equimult^s G, H;
then,

Hyp. ∵ A is the same mult. of B,
that C is of D;
Constr. and that E is the same mult. of A
that F is of C;
1. 5. ∴ E is the same mult. of B,
that F is of D,
i. e. E, F are equimult^s of B, D:
Constr. but G, H are equimult^s of B, D:
∴ if E be a greater mult. of B than G is of B,
F is a greater mult. of D than H is of D:
i. e. if E be $>$ G
F is $>$ H:

in like manner
if E be $=$ or $<$ G,
it may be shown that
F is $=$ or $<$ H:

Constr but E, F are any equimult^s of A, C;
G, H any equimult^s of B, D;
Def.5.5. and ∴ A : B :: C : D.

Next, let A the 1st be the same part of B the 2nd, that C the 3rd is of D the 4th:



in this case also,

$$A : B :: C : D.$$

For,

$\because$ A is the same part of B that C is of D.

$\therefore$ B is the same mult. of A that D is of C:

whence, by the preceding case,

$$B : A :: D : C;$$

$\therefore$ invly $A : B :: C : D.$ B. 5.

$\therefore$ *if the first be the same multiple, &c.*

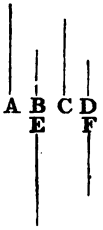
[Q. E. D.]

PROP. D. THEOR.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Let $A : B :: C : D$: and first let A be a mult. of B: C shall be the same mult. of D.

Take $E = A$, and whatever mult. A or E is of B, make F the same mult. of D: then,



$\because A : B :: C : D;$ Hyp.

and of B the 2nd, and D the 4th, equimult[s] E, F have been taken;

$\therefore A : E :: C : F:$ Cor.4.5.

but $A = E$, Constr.

$\therefore \quad C = F:$ Ax. 5.

Constr. and F is the same mult. of D that A is of B:
∴ C is the same mult. of D that A is of B.

See the last fig. in C. Next, let A be a part of B: C shall be the same part of D.

Hyp. ∵ A : B :: C : D,
B. 5. ∴ invˡʸ B : A :: D : C:
Hyp. but A is a part of B, *i.e.* B is a mult. of A;
∴ by preceding case, D is the same mult. of C;
i.e. C is the same part of D, that A is of B.

∴ *if the first, &c.* [Q. E. D.]

PROP. VII. THEOR.

Equal magnitudes have the same ratio to the same magnitude: and the same has the same ratio to equal magnitudes.

Let A, B be equal magnˢ, and C be any other, A and B shall each of them have the same rº to C: and C shall have the same rº to each of the magnˢ A, B.

Take of A, B any equimultˢ D, E; and of C any mult. F: then,

Constr. ∵ D is the same mult. of A, that E is of B,
Hyp. and that $A = B$;
Ax. 1.5. ∴ $D = E$;
∴ as D is $>, =$ or $<$ F,
so E is $>, =$ or $<$ F:

but D, E are any equimult^s of A, B, Constr
and F is any mult. of C;
$\therefore$ A : C :: B : C. Def.5.5

Likewise, C shall have the same r° to A, that it has to B. For, having made the same construction, it may in like manner be shown that

$$D = E;$$

and $\therefore$ as F is $>$, $=$ or $<$ D,
so it is $>$, $=$ or $<$ E:
but F is any mult. of C,
and D, E are any equimult^s of A, B;
$\therefore$ C : A :: C : B. Def.5.5.

$\therefore$ *equal magnitudes, &c.* [Q. E. D.]

PROP. VIII. THEOR.

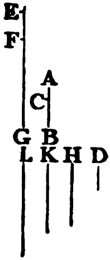
Of two unequal magnitudes the greater has a greater ratio to any other magnitude than the less has: and the same magnitude has a greater ratio to the less of two other magnitudes, than it has to the greater.

Let AB, BC be two unequal magn^s, AB $>$ BC; and let D be any other magn.: AB shall have a greater r° to D than BC has to D: and D shall have a greater r° to BC than it has to AB.

If the magn. w^h is not the greater of the two AC, CB, be $\nless$ D, take EF, FG the doubles of

AC, CB, as in Fig. 1. But if that w^h is not the greater of the two AC, CB, be $<$ D (as in Fig^s 2 and 3) this magn. whether it be AC or CB, can be multiplied, so as to become $>$ D.

Fig. 1.



Let it be so multiplied, and let the other be multiplied as often; and let EF be the mult. thus taken of AC, and FG the same mult. of CB: then,

EF, FG are each $>$ D:

and in every one of the cases, take H the double of D, K its triple, and so on, till the mult. of D be that w^h first becomes $>$ FG: let L be that mult. of D w^h is first $>$ FG, and K the mult. of D w^h is next $<$ L. Then,

$\because$ L is that mult. of D w^h first becomes $>$ FG

$\therefore$ the next preceding mult. K is $\not>$ FG;

i.e. FG is $\not<$ K:

and

Constr. $\because$ EF is the same mult. of AC, that FG is of CB,

I. 5. $\therefore$ FG is the same mult. of CB, that EG is of AB,

i.e. EG, FG are equi-mult^s of AB, CB:

and since it was shown that FG is $\not<$ K,

and, by the construction, EF is $>$ D;

$\therefore$ the whole EG is $>$ (K + D):

Constr. but (K + D) = L;

$\therefore$ EG is $>$ L:

Constr. but FG is $\not>$ L;

Fig. 2.

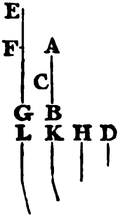
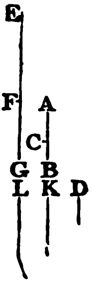


Fig. 3.



and it was proved that

EG, FG are equimult^s of AB, BC;

and L is a mult. of D; Constr.

$\therefore$ AB has to D a greater r^o than BC has to D. Def.7.5.

Also

D shall have to BC a greater r^o than it has to AB.

For, having made the same construction, it may be shown, in like manner, that

L is $>$ FG, but is $\ngtr$ EG:

and L is a mult. of D; Constr.

and FG, EG were proved to be equimult^s of CB, AB;

$\therefore$ D has to CB a greater r^o than it has to AB. Def.7.5.

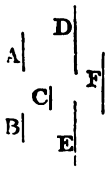
$\therefore$ *of two unequal magnitudes, &c.*

[Q. E. D.]

PROP. IX. THEOR.

Magnitudes which have the same ratio to the same magnitude are equal to one another: and those to which the same magnitude has the same ratio are equal to one another.

Let A, B have each of them the same r^o to C; then shall $A = B$.



For, if they are unequal, one must be $>$ the other: let A be $>$ B: then, by what was shown in the preceding proposition, there are some equimult^s of A, B, and some mult. of C such that the mult. of A is $>$ the mult. of B, but the mult. of B is $\ngtr$ that of C.

Let these mult^s be taken: and let, D, E be the equimult^s of A, B, and F the mult. of C, such that D may be $>$ F, but E $\ngtr$ F: then,

$\because$ A : C :: B : C,

and of A, B are taken equimult^s D, E,

and of C is taken a mult. F;

and that D is $>$ F;

$\therefore$ also E is $>$ F:

but E is $\ngtr$ F;

ef.5.5. w^h is impossible:

onstr. $\therefore$ A is not $\neq$ B,

i. e. A $=$ B.

Next, let C have the same r^o to each of the mag^s A, B: then shall A $=$ B.

For, if A $\neq$ B, one must be $>$ the other: let A be $>$ B: then, as was shown in Prop. 8th,

there is some mult. F of C,

and some equimult^s E, D of B, A such, that

F is $>$ E, but $\ngtr$ D:

and $\because$ C : B :: C : A,

Hyp. and that

F the mult. of the 1st is $>$ E the mult. of the 2nd,

Def.5.5. $\therefore$ F the mult. of the 3rd is $>$ D the mult. of the 4th:

but F is $\ngtr$ D;

Constr. w^h is impossible:

$\therefore$ A $=$ B. [Q. E. D.]

$\therefore$ *magnitudes which, &c.*

PROP. X. THEOR.

That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

Let A have to C a greater r° than B has to C: A shall be > B.

For,

∵ A has to C a greater r° than B has to C,

∴ there are some equimult^s of A, B, Def.7.5.
and some mult. of C such, that the mult. of A is
> the mult. of C, but the mult. of B is ≯ it:

let these mult^s be taken; and let D, E be the equimult^s of A, B, and F the mult. of C such, that

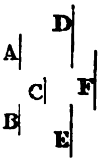
D is > F, but E is ≯ F:

then, D is > E;

and ∵ D, E are equimult^s of A, B,

and that D is > E;

∴ A is > B. Ax. 4.5.



Next, let C have a greater r° to B than it has to A: B shall be < A.

For, ∵ there is some mult. F of C, Def.7.5.
and some equimult^s E, D of B, A, such that

F is > E, but ≯ D:

∴ E is < D:

and ∵ E, D are equimult[s] of B, A,
and that E is < D,
Ax. 4.5. ∴ B is < A.

∴ *that magnitude, &c.* [Q. E. D.]

PROP. XI. THEOR.

Ratios that are the same to the same ratio, are the same to one another.

Let A : B :: C : D,
and also E : F :: C : D:
then shall A : B :: E : F.

G———— H———— K————
A—— C—— E——
B—— D—— F——
L———— M———— N————

Take of A, C, E, any equimult[s] whatever G, H, K; and of B, D, F, any whatever, L, M, N.

Then, ∵ A : B :: C : D,
and G, H are equimult[s] of A, C; and L, M, of B, D;
Def. 5.5. ∴ as G is >, = or < L,
so H is >, = or < M.

Again, ∵ C : D :: E : F,
and H, K, are equimult[s] of C, E; and M, N, of D, F;
∴ as H is >, = or < M,
so K is >, = or < N:

but it has been shown that

as G is $>, =$ or $<$ L,
so H is $>, =$ or $<$ M;
and ∴ as G is $>, =$ or $<$ L,
so K is $>, =$ or $<$ N;

and G, K are any equimult[s] whatever of A, E,
L, N any of B, F:

and ∴ $A : B :: E : F$. Def. 5. 5.

∴ *ratios that, &c.* [Q. E. D.]

PROP. XII. THEOR.

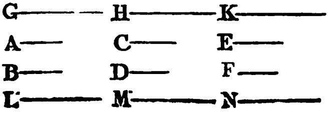
If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let any n° of magn[s] A, B, C, D, E, F be ::[ls],
i. e. $A : B :: C : D :: E : F$:

then shall

$$A : B :: A+C+E : B+D+F.$$

Take of A, C, E any equimult[s] whatever G, H, K;
and of B, D, F any whatever L, M, N:



then, ∵ $A : B :: C : D :: E : F$,

and that G, H, K are equimult^s of A, C, E,

and L, M, N equimult^s of B, D, F,

∴ as G is $>, =$ or $< L$,

ef.5.5. so H is $>, =$ or $< M$.

and also K is $>, =$ or $< N$,

and ∴ as G is $>, =$ or $< L$,

so $G+H+K$ is $>, =$ or $< L+M+N$:

but, if there be any n° of magn^s equimult^s of as many, each of each, whatever mult. one of them is

5. of its part, the same mult. is the whole of the whole;

∴ G and $G+H+K$ are any equimult^s of A, and $A+C+E$:

for the same reason,

L, and $L+M+N$ are any equimult^s of B, and $B+D+F$:

ef.5.5. ∴ $A : B :: A+C+E : B+D+F$.

∴ *if any number, &c.* [Q. E. D.]

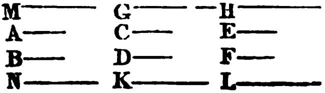
PROP. XIII. THEOR.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let A the 1^st have the same r° to B the 2^nd w^h C the 3^rd has to D the 4^th; but C the 3^rd a greater r° to D the 4^th, than E the 5^th has to F the 6^th: also

A the 1st shall have to B the 2nd a greater rº than E the 5th has to F the 6th.

For, $\because$ C has a greater rº to D, than E to F,
$\therefore$ there are some equimults of C, E, Def.7.5.
and some of D, F,
such that the mult. of C is $>$ the mult. of D,
but the mult. of E is $\not>$ that of F;



let these be taken; and let G, H be equimults of C, E, and K, L equimults of D, F such that
G may be $>$ K, but H $\not>$ L;
also, whatever mult. G is of C,
take M the same mult. of A;
and whatever mult. K is of D,
take N the same mult. of B:
then, $\because$ A : B :: C : D, Hyp.
and of A and C, M and G are equimults;
and of B and D, N and K are equimults:
$\therefore$ as M is $>$, $=$ or $<$ N,
so G is $>$, $=$ or $<$ K: Def.5.5.
but G is $>$ K; Constr.
$\therefore$ M is $>$ N:
but H is $\not>$ L: Constr.
and M, H are equimults of A, E;
and N, L are equimults of B, F;
$\therefore$ A has a greater rº to B, than E has to F. Def.7.5.

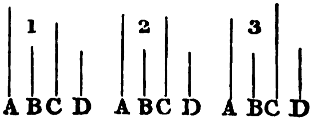
$\therefore$ *if the first, &c.* [Q. E. D.]

Cor.—And if the 1^{st} have a greater r^o to the 2^{nd} than the 3^{rd} has to the 4^{th}, but the 3^{rd} the same r^o to the 4^{th}, w^h the 5^{th} has to the 6^{th}; it may, in like manner, be dem^d that the 1^{st} has a greater r^o to the 2^{nd}, than the 5^{th} has to the 6^{th}.

PROP. XIV. THEOR.

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let A the 1^{st} have the same r^o to B the 2^{nd} w^h C the 3^{rd} has to D the 4^{th}: if A be $>$ C, B shall be $>$ D.



For,

$\because$ A is $>$ C, and B is any other magn.,

8. 5. $\therefore$ A has to B a greater r^o than C has to B:

Hyp. but, $A : B :: C : D$;

13. 5. $\therefore$ also C has to D a greater r^o than C has to B:

but of two $magn^s$, that to w^h the same magn. has

10. 5. the greater r^o is the less:

$\therefore$ D is $<$ B,

i. e. B is $>$ D.

Secondly, let $A = C$: then shall $B = D$.

For, $A : B :: C$ *i.e.* $A : D$;
and $\therefore B = D$. 9. 5

Thirdly, if A be $<$ C, B shall be $<$ D.

For, $C : D :: A : B$,
and C is $>$ A;
$\therefore$ by the first case,
D is $>$ B,
i. e. B is $<$ D.

$\therefore$ *if the first, &c.* [Q. E. D.]

PROP. XV THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

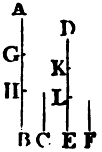
Let AB be the same mult. of C, that DE is of F: then shall $C : F :: AB : DE$.

For,
$\because$ AB is the same mult. of C, that DE is of F,
$\therefore$ there are as many magn^s in AB, each $= C$, as there are in DE, each $= F$:

let AB be div^d into magn^s, each $= C$, viz. AG, GH, HB:

and DE into magn^s, each $= F$, viz. DK, KL, LE: then the n^o of the first magn^s $=$ the n^o of the last:

and $\because$ $AG = GH = HB$,
and also $DK = KL = LE$;
7. 5. $\therefore AG : DK :: GH : KL$,
and $:: HB : LE$:



but as one antecedent is to its conse-
12. 5. quent, so are all the antecedents together to all the consequents together;

and $\therefore AG : DK :: AB : DE$:
but $AG = C$, and $DK = F$;
$\therefore C : F :: AB : DE$.

$\therefore$ *Magnitudes, &c.* [Q. E. D.]

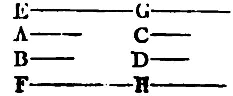
PROP. XVI. THEOR.

If four magnitudes of the same kind *be proportionals, they shall also be proportionals when taken alternately.*

Let A, B, C, D be four magn^{s} of the same kind, w^{h} are $::^{ls}$, viz. $A : B :: C : D$:
they shall also be $::^{ls}$, when taken alt^{ly}, viz.
$A : C :: B : D$.

Take of A, B, any equimult^{s} whatever, E, F;
and of C, D, any equimult^{s} whatever G, H:

then, $\because$ E is the same mult. of A, that F is of B, and that magn^{s} have the same r^{o} to one another w^{h}
15. 5. their equimult^{s} have;
$\therefore A : B :: E : F$;
Hyp. but $A : B :: C : D$;
11. 5. $\therefore C : D :: E : F$:



Again, ∵ G, H are equimult^s of C, D,
$\therefore$ C : D :: G : H: 15. 5.
but it was proved that
C : D :: E : F;
and $\therefore$ E : F :: G : H. 11. 5.

But when four magn^s are ::^{ls},
as the 1st is $>$, $=$ or $<$ the 3rd,
so the 2nd is $>$, $=$ or $<$ the 4th; 14. 5.
$\therefore$ as E is $>$, $=$ or $<$ G,
so F is $>$, $=$ or $<$ H;
and E, F are any equimult^s whatever of A, B;
G, H any whatever of C, D; Constr
$\therefore$ A : C :: B : D. Def. 5.5

If, then, four magnitudes, &c. [Q. E. D.]

PROP. XVII. THEOR.

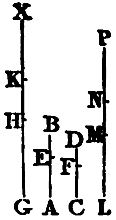
If magnitudes taken jointly be proportionals, they shall also be proportionals when taken separately: that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

Let AB, BE, CD, DF be the magn^s taken jointly, w^{h} are ::^{ls}, *i. e.* AB : BE :: CD : DF:
they shall also be ::^{ls} taken separately, viz.
AE : EB :: CF : FD.

Take of AE, EB, CF, FD any equimult
GH, HK, LM, MN ;
and again, of EB, FD take any equimult^s KX, NP
then

∵ GH is the same mult. of AE, that HK is of EB,
1 5. ∴ GH is the same mult. of AE, that GK is of AB:
but GH is the same mult. of AE, that LM is of CF ;
∴ GK is the same mult. of AB, that LM is of CF ;

Again,
∵ LM is the same mult. of CF, that MN is of FD ;
1 5. ∴ LM is the same mult. of CF, that LN is of CD :
but it was shown that
LM is the same mult. of CF,
that GK is of AB,
and ∴ GK is the same mult. of AB,
that LN is of CD ;
i. e. GK, LN are equimult^s of AB, CD.



Next,
∵ HK is the same mult. of EB,
that MN is of FD ;
and also,
KX is the same mult. of EB, that NP is of FD ;
2. 5. ∴ HX is the same mult. of EB, that MP is of FD.

Hyp. And ∵ AB : BE :: CD : DF,
and that GK, LN are equimult^s of AB, CD,
and HX, MP are equimult^s of EB, FD:
Def.5.5 ∴ as GK is $>$, $=$ or $<$ HX,
so LN is $>$, $=$ or $<$ MP :
but if GH be $>$ KX,
then, the com. part HK being added to both,
Ax.4.1. GK is $>$ HX ;
∴ also, LN is $>$ MP ;

and, MN being taken away from both,
LM is $>$ NP : Ax. 5.]
$\therefore$ if GH be $>$ KX,
LM is $>$ NP :
And in like manner it may be dem[d] that
if GH be $=$ KX, or be $<$ KX,
also LM is $=$ NP, or is $<$ NP :
but GH, LM are any equimult[s] of AE, CF Constr.
and KX, NP are any of EB, FD ; Def.5.5.
$\therefore$ AE : EB :: CF : FD.

If, then, magnitudes, &c. [Q. E. D.]

PROP. XVIII. THEOR.

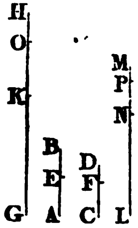
If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly: that is, if the first be to the second, as the third to the fourth, the first and second together shall be to the second, as the third and fourth together, to the fourth.

Let AE, EB, CF, FD be ::[ls] ;
i. e. AE : EB :: CF : FD :
they shall also be ::[ls] when taken jointly ;
i. e. AB : BE :: CD : DF.

Take of AB, BE, CD, DF any equimult[s] whatever GH, HK, LM, MN ; and again, of BE, DF, take any equimult[s] whatever KO, NP :
then, $\because$ KO, NP, are equimult[s] of BE, DF,
and that KH, NM are also equimult[s] of BE, DF ;

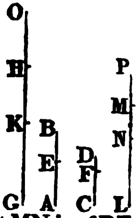
∴ if KO, the mult. of BE, be $>$ KH,
wh is a mult. of the same BE,
then NP, the mult. of DF, is also $>$ NM,
the mult. of the same DF;
and if KO be $=$ KH, or be $<$ KH,
also NP is $=$ NM, or is $<$ NM.

First, let KO be $\not>$ KH;
∴ NP is $\not>$ NM;
and ∵ GH, HK are equimult[s] of
AB, BE, and that AB is $>$ BE,
Ax 3. 5 ∴ GH is $>$ HK;
but KO is $\not>$ HK;
∴ GH is $>$ KO.



In like manner it may be shown
that LM is $>$ NP:
∴ if KO be $\not>$ KH,
then GH, the mult. of AB, is always $>$ KO,
the mult. of BE;
and likewise,
LM, the mult. of CD, is $>$ NP, the mult. of DF

Next, let KO be $>$ KH; then, as has been shown
NP is $>$ NM:
and ∵ the whole GH is the same
mult. of the whole AB,
that HK is of BE,
∴ the rem[r] GK is the same mult.
5. 5. of the rem[r] AE that GH is of AB;
wh is the same that LM is of CD.



In like manner,
∵ LM is the same mult. of CD, that MN is of DF
∴ the rem[r] LN is the same mult. of the rem[r] CF
5. 5. that the whole LM is of the whole CD:

but it was shown that

LM is the same mult. of CD, that GK is of AE;

$\therefore$ GK is the same mult. of AE, that LN is of CF;

i. e. GK, LN are equimult[s] of AE, CF.

And $\because$ KO, NP are equimult[s] of BE, DF,

$\therefore$ if from KO, NP there be taken KH, NM,

w[h] are likewise equimult[s] of BE, DF,

the rem[rs] HO, MP are either $=$ BE, DF, 6. 5.

or are equimult[s] of them.

First, let HO, MP be $=$ BE, DF:

then, $\because$ AE : EB :: CF : FD, Hyp.

and that GK, LN are equimult[s] of AE, CF:

$\therefore$ GK : EB :: LN : FD: Cor.4.5.

but HO $=$ EB, and MP $=$ FD:

$\therefore$ GK : HO :: LN : MP.

$\therefore$ as GK is $>$, $=$ or $<$ HO,

so LN is $>$, $=$ or $<$ MP. A. 5.

But let HO, MP be equimult[s] of EB, FD:

then, $\because$ AE : EB :: CF : FD, Hyp.

and that of AE, CF are taken equimult[s] GK, LN;

and of EB, FD the equimult[s] HO, MP,

$\therefore$ as GK is $>$, $=$ or $<$ HO,

so LN is $>$, $=$ or $<$ MP: Def.5.5

w[h] was likewise shown in the preceding case.

But, if GH be $>$ KO,

taking KH from both,

GK is $>$ HO; Ax.5.1.

$\therefore$ also LN is $>$ MP:

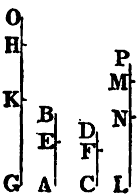
and $\therefore$ adding NM to both,

LM is $>$ NP. Ax.4.1.

In like manner it may be shown,

that if GH be $=$ KO, or be $<$ KO,

also LM is $=$ NP, or is $<$ NP.



And in the case in w^{h} KO is $\ngtr$ KH, it has been shown that

GH is always $>$ KO, and likewise LM $>$ NP:

Constr. but GH, LM are any equimult^{s} of AB, CD, and KO, NP are any whatever of BE, DF;

Def.5.5. $\therefore$ AB : BE :: CD : DF.

If then, magnitudes, &c. [Q. E. D.]

PROP. XIX. THEOR.

If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall be to the remainder as the whole to the whole.

From the magn^{s} AB, CD let the parts AE, CF be taken such that

the whole AB : the whole CD :: AE : CF: then shall

the rem^{r} EB : the rem^{r} FD :: AB : CD.

For, $\because$ AB : CD :: AE : CF,

16. 5. $\therefore$ alt^{ly} AB : AE :: CD : CF:

but, if magn^{s} taken jointly be ::^{ls},

17. 5. they are also ::^{ls}, taken separately;

and $\therefore$ BE : AE :: DF : CF:

and alt^{ly} BE : DF :: AE : CF:

but, by hyp., AE : CF :: AB : CD;

1. 5. and $\therefore$ the rem^{r} BE : rem^{r} DF :: AB : CD.

$\therefore$ *if the whole, &c.* [Q. E. D.]

Cor.—If the whole be to the whole, as a magn. taken from the first, is to a magn. taken from the other; the rem^{r} shall likewise be to the rem^{r}, as the magn. taken from the first to that taken from the other. The demonstration is contained in the preceding.

PROP. E. THEOR.

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let AB : BE :: CD : DF:
then shall AB : AE :: CD : CF.

For,
∵ AB : BE :: CD : DF,
∴ by div^{n}, AE : BE :: CF : DF; 17. 5.
and by inv^{n}, BE : AE :: DF : CF; B. 5.
∴ by comp^{n}, AB : AE :: CD : CF. 18. 5.

∴ *If four, &c.* [Q. E. D.]

PROP. XX. THEOR.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then if the first be greater than the third, the fourth

shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three magn[s], and D, E, F other three, w[h], taken two and two, have the same r[o], viz.

$A : B :: D : E$

and $B : C :: E : F$:

as A is $>, =$ or $<$ C,

so shall D be $>, =$ or $<$ F.

First, let A be $>$ C: then,

$\because$ B is any other magn.

8. 5. and that the greater has to the same magn. a greater ratio than the less has to it;

$\therefore$ A has to B a greater r[o] than C to B:

Hyp. but $D : E :: A : B$,

13. 5. $\therefore$ D has to E a greater r[o] than C to B:

and $\because$ $B : C :: E : F$,

B. 5. $\therefore$ inv[ly], $C : B :: F : E$:

and it was shown that

D has to E a greater r[o] than C has to B:

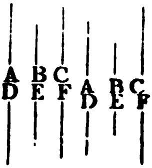
Cor. 13. 5. $\therefore$ D has to E a greater r[o] than F has to E:

10. 5. but the magn. w[h] has a greater r[o] than another to the same magn. is the greater of the two;

$\therefore$ D is $>$ F.

Secondly, let $A = C$: then shall $D = F$.

For,



$\because$ $A = C$,

7. 5. $\therefore A : B :: C : B$.

Hyp. but $A : B :: D : E$;

Hyp. & B. 5. and $C : B :: F : E$;

11. 5. $\therefore D : E :: F : E$,

9. 5. and $\therefore$ $D = F$.

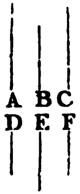
Next, let A be < C: D shall be < F.
For, C is > A:
and, as was shown in the first case,
C : B :: F : E,
and also B : A :: E : D;
∴ by the first case, F is > D,
i. e. D is < F.

∴ *if there be three, &c.* [Q. E. D.]

PROP. XXI. THEOR.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

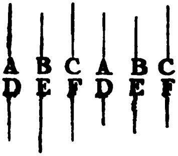
Let A, B, C be three magn^s and D, E, F other three, w^h have the same r^o taken two and two, but in a cross order, viz.



A : B :: E : F,
and B : C :: D : E:
as A is >, = or < C,
so shall D be >, = or < F.
First, let A be > C: then,
∵ B is any other magn.,
∴ A has to B a greater r^o than C to B: 8. 5.
but E : F :: A : B; Hyp.
∴ E has to F a greater r^o than C to B: 13. 5.

Hyp. and $\because$ $B : C :: D : E$,
$\therefore$ inv^{ly} $C : B :: E : D$:
and it was shown that,
E has to F a greater r° than C has to B;
Cor. 13. 5. $\therefore$ E has to F a greater r° than E has to D:
but the magn. to w^{h} the same has a greater r° than
10. 5. it has to another, is the less of the two:
$\therefore$ F is $< D$.
i. e. D is $> F$.

Secondly, let $A = C$: then shall $D = F$.



For, $\because$ $A = C$,
7. 5. $\therefore$ $A : B :: C : B$:
Hyp. but $A : B :: E : F$;
and $C : B :: E : D$;
11. 5. $\therefore$ $E : F :: E : D$,
9. 5. and $\therefore$ $D = F$.

Next, let A be $< C$:
D shall be $< F$.

For C is $> A$; and, as was shown,
$C : B :: E : D$,
and also $B : A :: F : E$;
$\therefore$ by case first, F is $> D$,
i. e. D is $< F$.

$\therefore$ *if there be three, &c.* [Q. E. D.]

PROP. XXII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have

the same ratio; the first shall have to the la of the first magnitudes the same ratio which th first has to the last of the others.

N.B. This is usually cited by the words, "*e æquali*," or "*ex æquo*."

First, let there be three magn^s A, B, C, and many others, D, E, F, w^h, taken two and tw have the same r^o, *i. e.* such that

$$A : B :: D : E;$$

and $B : C :: E : F$:

then shall $A : C :: D : F$.

Take of A, D any equimult^s whatever G, H of B, E any whatever K, L; and of C, F any whatever M, N; then,

$\because$ $A : B :: D : E$, and that G, H are equimult^s of A, D, and K, L equimult^s of B, E;

$\therefore$ $G : K :: H : L$:

for the same reason,

$K : M :: L : N$:

and,

$\because$ there are three magn^s G, K, M, and other thr H, L, N, w^h, taken two and two, have the same r

$\therefore$ as G is $>, =$ or $<$ M,

so H is $>, =$ or $<$ N:

but G, H are any equimult^s whatever of A, D and M, N are any equimult^s whatever of C, F

$\therefore$ $A : C :: D : F$.

Next, let there be four magn^s A, B, C, D, an other four E, F, G, H, w^h, taken two and two, hav the same r^o vis.

$$\begin{aligned} A &: B :: E : F, \\ B &: C :: F : G, \\ \text{and } C &: D :: G : H : \\ \text{then shall } A &: D :: E : H. \end{aligned}$$

A.B.C.D.
E.F.G.H.

For,

∵ A, B, C are three magn[s], and E, F, G, other three, w[h], taken two and two, have the same r[o];

∴ by the foregoing case,

$$A : C :: E : G;$$

$$\text{but } C : D :: G : H;$$

∴ again, by the first case,

$$A : D :: E : H;$$

and so on, whatever be the n[o] of magn[s].

∴ *if there be any number, &c.* [Q. E. D.]

PROP. XXIII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others.

N.B. This is usually cited by the words "*ex æquali in proportione perturbatâ;*" or "*ex æquo perturbato.*"

First, let there be three magn[s] A, B, C, and other

three D, E, F, w^{h}, taken two and two in a cro
order, have the same r^{o},

viz. $A : B :: E : F$,
and $B : C :: D : E$:
then shall $A : C :: D : F$.

Take of A, B, D, any equimult^{s} whatever G, H, K
and of C, E, F, any equimult^{s} whatever L, M, N

then,

$\because$ G, H are equimult^{s} of A, B,
and that magn^{s} have the same
r^{o} w^{h} their equimult^{s} have;
$\therefore A : B :: G : H$:
and, for the same reason,
$E : F :: M : N$:
but $A : B :: E : F$;
$\therefore G : H :: M : N$:
and $\because B : C :: D : E$;
and that H, K are equimult^{s} of B, D,
and L, M, of C, E;
$\therefore H : L :: K : M$:

and it has been shown that
$G : H :: M : N$:

hence,
$\because$ there are three magn^{s} G, H, L, and other thr
K, M, N, w^{h}, taken two and two in a cross orde
have the same r^{o};
$\therefore$ as G is $>, =$ or $<$ L,
so K is $>, =$ or $<$ N:

but G, K are any equimult^{s} whatever of A, D;
and L N are any whatever of C, F;
$\therefore A : C :: D : F$.

Next, let there be four magn^s A, B, C, D, and other four E, F, G, H, w^h, taken two and two in a cross order, have the same r^o,

viz. A : B :: G : H:
B : C :: F : G;
and C : D :: E : F:
then shall A : D :: E : H.

A.B.C.D
E.F.G.H

For,

∵ A, B, C are three magn^s,
and F, G, H other three, w^h,
taken two and two in a cross order, have the same r^o,
∴ by the 1st case,
A : C :: F : H:
but C : D :: E : F;
∴ again, by the 1st case,
A : D :: E : H:
and so on, whatever be the n^o of the magn^s.

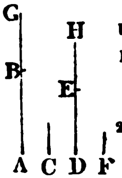
∴ *if there be any number, &c.* [Q. E. D.]

PROP. XXIV. THEOR.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let AB the 1st have to C the 2nd, the same r^o w^h DE the 3rd has to F the 4th;

and let BG the 5th have to C the 2nd, the same r° wh EH the 6th has to F the 4th:
AG (the 1st + the 5th) shall have to C the 2nd the same r° wh DH (the 3r + the 6th) has to F the 4th.



For,
∵ BG : C :: EH : F, Hyp.
∴ invly C : BG :: F : EH: B. 5.
and ∵ AB : C :: DE : F,
and C : BG :: F : EH;
∴ *ex æq.* AB : BG :: DE : EH: 22. 5.
and ∵ these magns are :: ls,
∴ they are also :: ls, taken jointly; 18. 5.
∴ AG : GB :: DH : EH:
but GB : C :: EH : F; Hyp.
∴ *ex æq.* AG : C :: DH : F. 22. 5.

∴ *if the first, &c.* [Q. E. D.]

Cor. 1.—If the same hypothesis be made as in the proposition, the excess of the 1st and 5th shall be to the 2nd, as the excess of the 3rd and 6th is to the 4th. The demn of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2.—The proposition holds true of two ranks of magns, whatever be their n°, of wh each of the first rank has to the 2nd magn. the same r° that the corresponding one of the second rank has to a 4th magn.; as is manifest.

PROP. XXV. THEOR.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

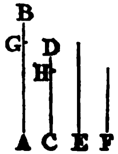
Let the four magns AB, CD, E, F be : : ls, viz

AB : CD : : E : F ;

and let AB be the greatest of them,

A. & 14. 5. and consequently F the least :

AB + F shall be > CD + E.



Take AG = E, and CH = F :

then, ∵ AB : CD : : E : F :

and that AG = E, CH = F,

7. & 11. 5. ∴ AB : CD : : AG : CH :

and

∵ the whole AB : the whole CD : : AG : CH,

19. 5. ∴ the remr GB : the remr HD : : AB : CD.

Hyp. but AB is > CD ;

A. 5. ∴ GB is > HD :

and ∵ AG = E, and CH = F,

Ax. 2. 1. ∴ AG + F = CH + E :

to the unequal magns GB, HD, of wh GB is the greater, let there be added these equal magns, viz. AG + F to GB, and CH + E to HD ;

Ax. i. 1. then AB + F is > CD + E.

∴ *if four magnitudes, &c.* [Q. E. D.]

PROP. F. THEOR.

Ratios which are compounded of the same ratios, are the same to one another.

Let A : B :: D : E,
and B : C :: E : F:

the r^{o} w^{h} is compounded of the r^{os} of A to B, and B to C, w^{h}, by the def^{n} of compound r^{o}, is the r^{o} of A to C, shall be the same with the r^{o} of D to F, w^{h}, by the same def^{n}, is compounded of the r^{os} of D to E, and E to F.

For,

∵ there are three magn^{s} A, B, C, and three others D, E, F, w^{h}, taken two and two, in order, have the same r^{o},

$\boxed{\begin{matrix}\text{A.B.C.}\\ \text{D.E.F.}\end{matrix}}$

∴ *ex æq.* A : C :: D : F. 22. 5.

Next,

let A : B :: E : F, and B : C :: D : E:

then, *ex æquali in proportione perturbatâ*,

$\boxed{\begin{matrix}\text{A.B.C.}\\ \text{D.E.F.}\end{matrix}}$

A : C :: D : F: 23. 5.

i. e. the ratio of A *to* C, *which is compounded of the ratios of* A *to* B, *and* B *to* C, *is the same with the ratio of* D *to* F, *which is compounded of the ratios of* D *to* E, *and* E *to* F. [Q. E. D.]

And in like manner the prop^{n} may be dem^{d} whatever be the n^{o} of r^{os} in either case.

PROP. G. THEOR.

If several ratios be the same to several ratios, each to each; the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

Let A : B :: E : F, and C : D :: G : H; also, A : B :: K : L, and C : D :: L : M; then, by the def^{n} of compound r^{o}, the r^{o} of K to M is compounded of the r^{os} of K to L, and L to M, w^{h} are the same with the r^{os} of A to B, and C to D.

ABCD	KLM
EFGH	NOP

Again,

let N : O :: E : F, and O : P :: G : H: then, the r^{o} of N to P is compounded of the r^{os} of N to O and O to P, w^{h} are the same with the r^{os} of E to F and G to H: and it is to be shown that the r^{o} of K to M is the same with the r^{o} of N to P; or that

K : M :: N : P.

Now,

∵ K : L :: (A : B *i.e.* :: E : F *i.e.*) :: N : O,
and L : M :: (C : D *i.e.* :: G : H *i.e.*) :: O : P,
22. 5. ∴ *ex æq.* K : M :: N : P.

∴ *if several ratios, &c* [Q. E. D.]

PROP. H. THEOR.

If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them; then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

Let the first r^{os} be those of
A to B, B to C, C to D, D to E, and E t F;
and let the other r^{os} be those of
G to H, H to K, K to L, and L to M:
also, let the r^{o} of A to F, w^{h} is compounded of the first r^{os}, be the same with the r^{o} of G to M, w^{h} is compounded of the other r^{os};

A.B.C.D.E.F
G.H.K.L.M.

and, besides, let the r^{o} of A to D, w^{h} is compounded of the r^{os} of A to B, B to C, C to D, be the same with the r^{o} of G to K, w^{h} is compounded of the r^{os} of G to H, and H to K:
then the r^{o} compounded of the rem^{g} first r^{os}, viz. of the r^{os} of D to E, and E to F, w^{h} compounded r^{o} is that of D to F, shall be the same with the r^{o} of K to M, w^{h} is compounded of the rem^{g} r^{os} of K to L and L to M of the other r^{os}.

For, $\because$ by hyp. $A : D :: G : K$,
3. 5. $\therefore$ by inv^n, $D : A :: K : G$;
Hyp. and by hyp. $A : F :: G : M$;
2. 5. $\therefore$ *ex æq.* $D : F :: K : M$.

$\therefore$ *if a ratio which is, &c.* [Q. E. D.]

PROP. K. THEOR.

If there be any number of ratios, and any number of other ratios such, that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be the same to one of the last ratios, or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios; then the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same, each to each, to the remaining ratios of the first, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are the same, each to each, to these remaining ratios.

Let A to B, C to D, E to F be the first r^{os}; and G to H, K to L, M to N, O to P, Q to R, the other r^{os}:

and let A : B :: S : T,
C : D :: T : V,
E : F :: V : X :

then, by the def^n of compound r^o, the r^o of S to X is compounded of the r^{os} of S to T, T to V, V to X, w^h are the same to the r^{os} of A to B, C to D, E to F, each to each.

	h, k, l.	
	A, B ; C, D ; E, F.	S, T, V, X.
G, H ;	K, L ; M, N ; O, P ; Q, R.	Y, Z, a, b, c, d.
e, f, g.	m, n, o, p.	

Also,
let G : H :: Y : Z, and K : L :: Z : a ;
M : N :: a : b ; O : P :: b : c ;
and Q : R :: c : d :

then, by the same def^n,
the r^o of Y to d is compounded of the r^{os} of Y to Z, Z to a, a to b, b to c, and c to d,
w^h are the same, each to each, to the r^{os} of G to H, K to L, M to N, O to P, and Q to R :
∴ by the hyp. S : X :: Y : d.

Also, let the r^o of A to B, *i. e.* the r^o of S to T, w^h is one of the first r^{os}, be the same to the r^o of e to g, w^h is compounded of the r^{os} of e to f and f to g, w^h, by the hyp., are the same to the r^{os} of G to H and K to L, two of the other r^{os} ; and let the r^o of h to l be that w^h is compounded of the r^{os} of h to k and k to l, w^h are the same to the rem^g first r^{os}, viz. C to D and E to F ; also, let the r^o of m to p be that w^h is compounded of the r^{os} m to n, n to o, and o to p, w^h are the same, each to each, to the rem^g other r^{os}, viz. M to N, O to P,

and Q to R: then the r° h to l shall be the same to the r° of m to p;

or h : l :: m : p.

		h, k, l.						
	A, B;	C D;	E, F.		S, T, V, X.			
G,H;	K,L;	M,N;	O,P;	Q,R;	Y,Z,a,b,c,d.			
e, f, g.		m, n, o, p.						

For,

∵ e : f :: (G : H *i. e.* ::) Y : Z;

and f : g :: (K : L *i. e.* ::) Z : a;

22. 5. ∴ *ex æq.* e : g :: Y : a:

and, by hyp.

e : g :: A : B *i. e.* :: S : T;

11. 5. ∴ S : T :: Y : a;

B. 5. and, by inv^{n}, T : S :: a : Y;

Hyp. but S : X :: Y : d;

22. 5. ∴ *ex æq.* T : X :: a : d;

Also,

Hyp. ∵ h : k :: (C : D *i. e.* ::) T : V;

and k : l :: (E : F *i. e.* ::) V : X;

∴ *ex æq.* h : l :: T : X:

in like manner it may be dem^{d} that

m : p :: a : d;

and it has been shown that

T : X :: a : d;

11. 5. ∴ h : l :: m : p.

[Q. E. D.]

The prop^{ns} G and K are usually, for the sake of brevity, expressed in the same terms with F and H: and therefore it was proper to show the true meaning of them when they are so expressed, especially as they are very frequently made use of *by geometers*.

BOOK VI.

DEFINITIONS.

I.

Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

II.

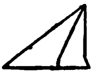
" Reciprocal figures, viz. triangles and parallelo-
" grams, are such as have their sides about two
" of their angles proportionate in such a manner
" that a side of the first figure is to a side of the
" other, as the remaining side of this other is to
" the remaining side of the first."

III.

A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

IV

The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.



PROP. I. THEOR.

Triangles and parallelograms of the same altitude are one to another as their bases.

Let the △ˢ ABC, ACD, and the ▱ˢ EC, CF have the same altit. viz. the ⊥ drawn from the pᵗ A to BD: then shall

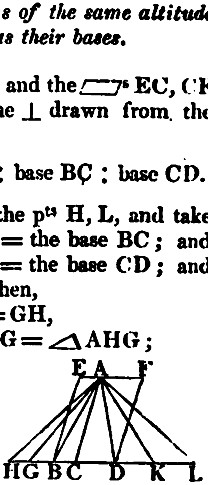
$$\left.\begin{array}{l}\triangle ABC : \triangle ACD \\ \text{and } \square EC : \square CF\end{array}\right\} :: \text{base } BC : \text{base } CD.$$

Prod. BD both ways to the pᵗˢ H, L, and take
S. 1. any nᵒ of |ˢ BG, GH, each = the base BC; and any nᵒ of |ˢ DK, KL, each = the base CD; and join AG, AH, AK, AL; then,

$$\because \quad CB = BG = GH,$$

38. 1. $$\therefore \triangle ABC = \triangle ABG = \triangle AHG;$$

and ∴ whatever mult. the base HC is of the base BC, the same mult. is △ AHC of △ ABC:



for the same reason, whatever mult. the base LC is of the base CD, the same mult. is △ ALC of △ ACD:

and, as base HC is >, = or < base CL,

38. 1 so △ AHC is >, = or < △ ALC:

Hence, ∵ there are four magnˢ, viz. the two bases BC, CD, and the two △ˢ ABC, ACD: and of the base BC, and the △ ABC, the 1ˢᵗ and 3ʳᵈ, any equimultˢ whatever have been taken, viz. the base HC and the △ AHC;

and of the base CD and the △ ACD, the 2nd and 4th,
have been taken any equimult[s] whatever, viz.
the base CL and the △ ALC;
and ∵ it has also been shown that
as the base HC is >, = or < CL,
so the △ AHC is >, = or < ALC:
∴ △ ABC : △ ACD :: base BC : base CD. Def. 5. 5.

And ∵ the ▱ CE is double of the △ ABC, 41. 1.
and the ▱ CF double of the △ ACD,
and that magn[s] have the same r[o] w[h] their 15. 5.
equimult[s] have;
∴ △ ABC : △ ACD :: ▱ CE : ▱ CF;
but also
△ ABC : △ ACD :: base BC : base CD;
and
∴ ▱ CE : ▱ CF :: base BC : base CD. 11. 5.

∴ *triangles, &c.* [Q. E. D.]

Cor.—From this it is plain that △[s] and ▱[s] of equal altit[s] are to each other as their bases.

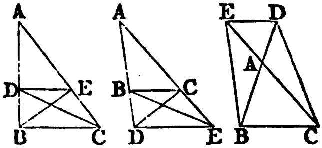
For, let the fig[s] be placed so as to have their bases in the same |; and draw ⊥[s] from the vertices to the bases: then,

∵ the ⊥[s] are = and ∥ to one another, 33. 1.
∴ the | w[h] joins the vertices is ∥ to that in w[h] their
bases are; 28. 1.
and, if the same const[n] be made as in the prop[n],
the dem[n] will be the same.

PROP. II. THEOR.

If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally: And if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Let DE be drawn ∥ BC, a side of the △ ABC: then shall BD : DA :: CE : EA.



Join BE, CD: then,

∵ the △s BDE, CDE are on the same base DE
and between the same ∥s DE, BC,

37. 1. ∴ △BDE = △CDE;

and

7. 5. ∴ these △s have the same r° to the same △ ADE,
or △ BDE : △ADE :: △CDE : △ ADE:
but ∵ the △s BDE, ADE have the same altit.,
viz. the ⊥ drawn from the p^t E to AB,

1. 6. ∴ △ BDE : △ ADE :: base BD : base DA:
and for the same reason,
△ CDE : △ ADE :: base CE : base EA:

11. 5. ∴ BD : DA :: CE : EA.

next, let the sides AB, AC of the △ ABC

or these sides prod^d, be cut $::^{ly}$ in the p^{ts} D, E, *i.e.* so that BD : DA :: CE : EA: then,

if | DE be drawn, DE shall be || BC.

For, the same constr^n being made,

$\because$ BD : DA :: CE : EA;

and BD : DA :: $\triangle$BDE : $\triangle$ADE, 1. 6.

and CE : EA :: $\triangle$CDE : $\triangle$ADE;

$\therefore$ $\triangle$ BDE : $\triangle$ ADE :: $\triangle$CDE : $\triangle$ADE, 11. 5.

i. e. the $\triangle^s$ BDE, CDE have the same r^o to the $\triangle$ADE:

$\therefore$ $\triangle$BDE = $\triangle$CDE: 9. 5.

and these $\triangle^s$ are on the same base DE:

but equal $\triangle^s$ on the same base are between the same $||^s$; 39. 1

and $\therefore$ DE is || BC.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. III. THEOR.

If the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another: and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.

Let the $\angle$ BAC of a $\triangle$ABC be bis^d by the | AD: then shall BD : DC :: BA : AC.

31. 1. Through the p^t C draw CE || DA, and let BA prod^d meet CE in E: then,

$\because$ the | AC meets the ||^s AD, EC,

29. 1. $\therefore \angle$ ACE = the alt. $\angle$ CAD:

but, by hyp. $\angle$ CAD = $\angle$ BAD:

Ax. 1. $\therefore \angle$ BAD = $\angle$ ACE.

Again,

$\because$ the | BAE meets the ||^s AD, EC,

29. 1. $\therefore \angle$ BAD = the int. and opp. $\angle$ AEC:

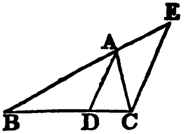
but, from above,

$\angle$ BAD = $\angle$ ACE,

Ax. 1. $\therefore \angle$ ACE = $\angle$ AEC,

and

6. 1. $\therefore$ side AE = side AC:



2. 6. and $\because$ AD is || EC one of the sides of the $\triangle$ BCE,

$\therefore$ BD : DC :: BA : AE:

but AE = AC;

7. 5. $\therefore$ BD : DC :: BA : AC.

Next, let BD : DC :: BA : AC, and join AD: the $\angle$ BAC shall be bis^d by the | AD.

For, the same constr^n being made,

$\because$ AD is || EC,

2. 6. $\therefore$ BD : DC :: BA : AE:

and, by hyp. BD : DC :: BA : AC;

11. 5 $\therefore$ BA : AE :: BA : AC;

9. 5. $\therefore$ AC = AE,

5. 1. and $\therefore \angle$ AEC = $\angle$ ACE:

but $\angle$ AEC = the ext. and opp. $\angle$ BAD

29. 1. and $\angle$ ACE = the alt. $\angle$ CAD;

Ax. 1. $\therefore \angle$ BAD = $\angle$ CAD,

i. e. the $\angle$ BAC is bis^d by the | AD.

$\therefore$ *if the angle*, &c. [Q. E. D.]

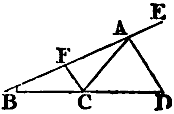
PROP. A. THEOR.

If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles, by a straight line which also cuts the base produced; the segments between the dividing line and the extremities of the base have the same ratio which the other sides of the triangle have to one another: and if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

Let the side BA of a $\triangle$ ABC be prod[d] to E; and let the ext[r] $\angle$ CAE be bis[d] by the | AD w[h] meets the base prod[d] in D:

then shall BD : DC :: BA : AC.

Through C draw CF $\parallel$ AD: 31. 1.



then,

$\because$ AC meets the $\parallel^s$ AD, FC,

$\therefore$ $\angle$ ACF = the alt. $\angle$ CAD: 29. 1.

but $\angle$ CAD = $\angle$ DAE; Hyp.

$\therefore$ $\angle$ DAE = $\angle$ ACF. Ax. 1.

Again,

$\because$ FAE meets the $\parallel^s$ AD, FC,

$\therefore$ the ext[r] $\angle$ DAE = the int. and opp. $\angle$ CFA; 29. 1.

but, from above,

$\angle$ ACF = $\angle$ DAE;

$\therefore$ also $\angle$ ACF = $\angle$ CFA; Ax. 1.

and $\therefore$ side AF = side AC · 6. 1.

and $\because$ AD is $\parallel$ FC, a side of the $\triangle$ BCF,

2. 6. $\therefore$ BD : DC :: BA : AF:

but AF = AC;

7. 5. $\therefore$ BD : DC :: BA : AC.

Next, let BD : DC :: BA : AC, and join AD: then shall $\angle$ CAD = $\angle$ DAE.

For, the same constr[n] being made,

$\because$ BD : DC :: BA : AC,

2. 6. and that also BD : DC :: BA : AF,

11. 5. $\therefore$ BA : AC :: BA : AF;

9. 5. $\therefore$ AC = AF,

5. 1. and $\angle$ AFC = $\angle$ ACF:

29. 1. but $\angle$ AFC = the ext[r] $\angle$ EAD

and $\angle$ ACF = the alt. $\angle$ CAD;

Ax. 1. $\therefore$ also $\angle$ EAD = $\angle$ CAD.

$\therefore$ *if the outward, &c.* [Q. E. D.]

PROP. IV. THEOR.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides. that is, are the antecedents or consequents of the ratios.

Let ABC, DCE be equiang[r] $\triangle$[s], having the $\angle$ ABC = $\angle$ DCE, $\angle$ ACB = $\angle$ DEC,

32. 1. & Ax. 3. and $\therefore$ $\angle$ BAC = $\angle$ CDE:

the sides about the equal $\angle$[s] of the $\triangle$[s] shall be :: [ls];

and those shall be the homol. sides w[h] are opp. to the equal $\angle^s$.

Let the $\triangle$ DCE be so placed, that its side CE may be contiguous to BC, and in the same | with it: 22. 1.

then, $\angle$ BCA $=$ $\angle$ CED; Hyp.

add to each $\angle$ ABC;

$\therefore \angle^s (ABC + BCA) = \angle^s (ABC + CED)$; Ax. 2.

but $\angle^s (ABC + BCA) <$ two r[t] $\angle^s$; 17. 1.

also $\angle^s (ABC + CED) <$ two r[t] $\angle^s$;

and $\therefore$ BA, ED, if prod[d], will meet: Ax. 12. 1.

let them be prod[d], and meet in the p[t] F:

then $\because$ $\angle$ ABC $=$ $\angle$ DCE,

$\therefore$ BF is $\parallel$ CD; 28. 1.

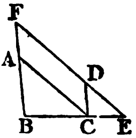
and $\because$ $\angle$ ACB $=$ $\angle$ DEC,

$\therefore$ AC is $\parallel$ FE: 28. 1.

$\therefore$ FACD is a $\square$;

and $\therefore$ AF $=$ CD, 34. 1.

AC $=$ FD.



And $\because$ AC is $\parallel$ FE, a side of the $\triangle$ FBE,

$\therefore$ AB : AF :: BC : CE: 2. 6.

but AF $=$ CD;

$\therefore$ AB : CD :: BC : CE: 7. 5.

and alt[y] AB : BC :: CD : CE.

Again, $\because$ CD is $\parallel$ BF,

$\therefore$ BC : CE :: FD : DE: 2. 6.

but FD $=$ AC;

$\therefore$ BC : CE :: AC : DE, 7. 5.

and alt[y] BC : AC :: CE : DE; 16. 5.

but, from above,

AB : BC :: DC : CE;

and $\therefore$ *ex æq.* AB : AC :: DC : DE. 22. 5.

$\therefore$ *the sides, &c.* [Q. E. D.]

PROP. V. THEOR.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular: and the equal angles shall be those which are opposite to the homologous sides.

Let the $\triangle^s$ ABC, DEF have their sides $::^{ls}$,

viz. AB : BC :: DE : EF,

BC : AC :: EF : FD;

and $\therefore$ *ex æq.* AB : AC :: DE : FD:

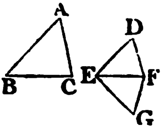
$\triangle$ ABC shall be equiang^r to $\triangle$ DEF,

and the $\angle^s$ w^h are opp. to the homol. sides shall be equal,

viz. $\angle$ ABC = DEF,

$\angle$ BCA = EFD,

and $\angle$ BAC = EDF.



At the p^{ts} E, F, in the | EF,

3 1. make

$\angle$ FEG = ABC, and $\angle$ EFG = BCA;

then is the rem^g $\angle$ BAC = the rem^g $\angle$ EGF,

32. 1. & Ax. 3. and $\therefore$ $\triangle$ ABC is equiang^r to $\triangle$ GEF:

4. 6. $\therefore$ the sides opp. to the equal $\angle^s$ are $::^{ls}$;

and $\therefore$ AB : BC :: GE : EF:

Hyp. but AB : BC :: DE : EF;

11. 5. $\therefore$ DE : EF :: GE : EF;

i. e. DE and GE have each the same r^o to EF;

9. 5. and $\therefore$ DE = GE:

sim^{ly} DF = FG:

hence, in the two $\triangle^s$ DEF, GEF,

$\because \begin{cases} \text{side DE = GE, EF is com.,} \\ \text{and base DF = base FG;} \end{cases}$

$\therefore$ the $\angle^s$ of one $\triangle$ = the $\angle^s$ of the other,

8. 1. each to each,

viz. $\angle$ DEF = GEF, DFE = GFE,
and EDF = EGF : 4. 1.
and $\because$ $\angle$ DEF = GEF,
and also $\angle$ GEF = ABC ; Constr.
$\therefore$ $\angle$ ABC = DEF : Ax. 1.
sim^{ly} $\angle$ ACB = DFE,
and $\angle$ BAC = EDF ;
$\therefore$ $\triangle$ ABC is equiang^{r} to $\triangle$ DEF.

$\therefore$ *if the sides, &c.* [Q. E. D.]

PROP. VI. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

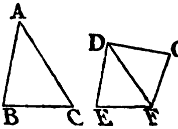
In the $\triangle^s$ ABC, DEF, let $\angle$ BAC = $\angle$ EDF, and also let the sides about these $\angle^s$ be :: ^{ls}, viz.

BA : AC :: ED : DF :

the $\triangle^s$ shall be equiang^{r}, and shall have

$\angle$ ABC = DEF, and ACB = DFE.

At the p^{ts} D, F, in the | DF, make
$\angle$ FDG = $\angle$ BAC or EDF; and $\angle$ DFG = ACB: 23. 1
then is the rem^{g} $\angle$ at B = the rem^{g} $\angle$ at G ; 32. 1. & Ax. 3.
and $\therefore$ $\triangle$ ABC is equiang^{r} to $\triangle$ DGF :
$\therefore$ BA : AC :: GD : DF : 4. 6.
but
BA : AC :: ED : FD ; Hyp.
$\therefore$ ED : DF :: GD : DF ; 11. 5.
and $\therefore$ ED = GD : 9. 5.



hence, in the △s EDF, GDF,

Constr. ∵ { side ED = DG, DF is com.,
and ∠ EDF = GDF,

∴ the rems ∠s = the rems ∠s, each to each,
viz. ∠ DFG = DFE, and DGF = DEF:

Constr. but ∠ DFG = ACB, and DGF = ABC;

Ax. 1. ∴ ∠ ACB = DFE, and ABC = DEF;

32. 1. & Ax. 3. and ∴ the rems ∠ BAC = the rems ∠ EDF:
∴ △ ABC is equiangr to △ DEF.

∴ *if two triangles, &c.* [Q. E. D.]

PROP. VII. THEOR.

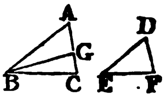
If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

In the two △s ABC, DEF, let one ∠ = one ∠, viz. ∠ BAC = EDF, and let the sides about two other ∠s ABC, DEF be ::ls, so that

AB : BC :: DE : EF:

and, first, let the rems ∠s at C, F be each < a rt ∠

△ ABC shall be equiangr to △ DEF, viz.
∠ ABC = DEF, and ∠ ACB = DFE.



For, if ∠ ABC be ≠ DEF, one must be > the other: let ABC be the greater; and at the pt B, in | AB, make

$\angle$ ABG = DEF : then, 23. 1.

$\because$ $\angle$ BAC = EDF, and $\angle$ ABG = DEF ; Hyp.

$\therefore$ the rem^{g} $\angle$ AGB = the rem^{g} $\angle$ DFE, 32. 1. &

and $\therefore$ $\triangle$ ABG is equiang^{r} to $\triangle$ DEF : Ax. 3.

$\therefore$ AB : BG :: DE : EF : 4. 6.

but, by hyp., AB : BC :: DE : EF :

$\therefore$ AB : BG :: AB : BC, 11. 5.

i. e. BG, BC have each the same r^{o} to AB,

and $\therefore$ BC = BG, 9. 5.

$\therefore$ $\angle$ BCG = BGC : 5. 1.

but, by hyp., $\angle$ BCG $<$ a r^{t} $\angle$;

$\therefore$ $\angle$ BGC $<$ a r^{t} $\angle$;

and $\therefore$ the adj^{t} $\angle$ AGB $>$ a r^{t} $\angle$: 13. 1.

but, from above, $\angle$ AGB = DFE ;

$\therefore$ $\angle$ DFE $>$ a r^{t} $\angle$:

but, by hyp., $\angle$ DFE $<$ a r^{t} $\angle$,

i. e. $\angle$ DFE is both $>$ and $<$ a r^{t} $\angle$:

w^{h} is absurd :

$\therefore$ $\angle$ ABC is not $\neq$ DEF,

i. e. $\angle$ ABC = DEF :

and $\angle$ BAC = EDF : Hyp.

$\therefore$ the rem^{g} $\angle$ ACB = the rem^{g} $\angle$ DFE, 32. 1. &

and $\triangle$ ABC is equiang^{r} to DEF. Ax. 3.

Next, let each of the $\angle$^{s} at C, F be $\not<$ a r^{t} $\angle$: in this case, $\triangle$ ABC shall be equiang^{r} to $\triangle$ DEF.

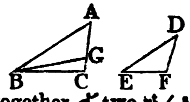
The same const^{n} being made, it may, in like manner, be proved that

BC = BG,

and $\therefore$ $\angle$ BCG = BGC :

but $\angle$ BCG $\not<$ a r^{t} $\angle$; Hyp

$\therefore$ $\angle$ BGC $\not<$ a r^{t} $\angle$,



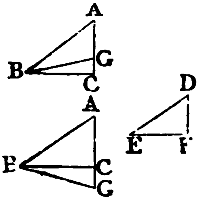
i. e. two $\angle$^{s} of $\triangle$ BGC are together $\not<$ two r^{t} $\angle$^{s} ;

w^{h} is impossible : 17. 1.

and $\therefore$ it may be proved, as in the 1st case, that

$\triangle$ ABC is equiang^r to DEF.

Lastly, let one of the $\angle^s$ at C, F, viz. the $\angle$ at C, be a r$^t\angle$: in this case also the $\triangle^s$ shall be equiang^r to each other.



For, if they be not, at the p^t B, in the | AB, make

$\angle$ ABG = DEF:

then it may be proved, as in the 1st case, that

BG = BC,

5. 1. and $\therefore$ $\angle$ BCG = BGC:

Hyp. but $\angle$ BCG is a r$^t\angle$;

Ax. 1. $\therefore$ also BGC is a r$^t\angle$;

and $\therefore$ two $\angle^s$ of $\triangle$ BGC are together $\not<$ two r$^t\angle^s$:

w^h is impossible:

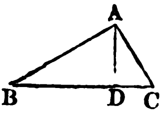
17. 1. $\therefore$ $\triangle$ ABC is equiang^r to DEF.

$\therefore$ *if two triangles, &c.* [Q. E. D.]

PROP. VIII. THEOR.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a r$^t\angle^d$ $\triangle$, having the r$^t\angle$ BAC: and from p^t A let AD be drawn $\perp$ to the base BC: the $\triangle^s$ ABD, ADC shall be sim^r to the whole $\triangle$ ABC, and to one another.



For, in the $\triangle^s$ ABC, ABD,

$\because \left\{ \begin{array}{l} \text{the r}^{t} \angle \text{ BAC} = \text{the r}^{t} \angle \text{ ADB,} \\ \text{and the } \angle \text{ at B is com. to both,} \end{array} \right.$ Ax. 1.1.

$\therefore$ the rem^{g} $\angle$ ACB = the rem^{g} $\angle$ BAD : 32. 1. & Ax. 3.

$\therefore$ the $\triangle$ ABC is equiang^{r} to $\triangle$ ABD,

and the sides about the equal $\angle^{s}$ are $::^{ls}$, 4. 6.

and $\therefore$ the $\triangle^{s}$ are sim^{r} : Def.1.6.

in the like manner it may be dem^{d} that

$\triangle$ ADC is equiang^{r} and sim^{r} to $\triangle$ ABC.

And,

$\because$ the $\triangle^{s}$ ABD, ACD are both equiang^{r} and sim^{r} to ABC,

$\therefore$ they are equiang^{r} and sim^{r} to each other.

$\therefore$ *in a right-angled triangle, &c.* [Q. E. D.]

Cor.—From this it is manifest that the $\perp$ drawn from the r^{t} $\angle$ of a r^{t} $\angle^{d}$ $\triangle$ to the base is a mean $::^{l}$ between the seg^{ts} of the base, and also that each of the sides is a mean $::^{l}$ between the base and the seg^{t} of it adj^{t} to that side:

for, in the $\triangle^{s}$ BDA, ADC,

BD : DA :: DA : DC ; 4. 6.

and in the $\triangle^{s}$ ABC, DBA,

BC : BA :: BA : BD ; 4. 6.

and also in the $\triangle^{s}$ ABC, ACD,

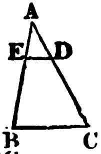
BC : CA :: CA : CD. 4. 6.

PROP. IX. PROB.

From a given straight line to cut off any part required.

Let AB be the given | ; it is req^{d} to cut off any part from it.

From the p^{t} A draw a | AC, making any $\angle$ with AB; and in AC take any p^{t} D, and take AC the same mult. of AD, that AB is of the part w^{h} is to be cut off from it; join BC, and draw DE || BC: AE shall be the part req^{d} to be cut off.



For,

∵ ED is || BC, a side of △ ABC,

2. 6. ∴ CD : DA :: BE : EA;

18. 5. and comp^{o}, CA : DA :: BA : EA:

Constr. but CA is a mult. of AD:

D. 5. and ∴ BA is the same mult. of AE:

∴ whatever part AD is of AC,

AE is the same part of AB.

∴ *from the straight line* AB *is cut off the part required.* [Q.E.F]

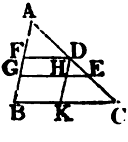
PROP. X. PROB.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the given divided straight line have.

Let AC be the given div^{d} |, and AB the | to be div^{d}: it is req^{d} to divide AB sim^{ly} to AC.

Let AC be div^{d} in the p^{ts} D, E; place AB, AC so as to contain

31. 1. any $\angle$, join BC, and through D, E draw DF, EG ||^{s} to BC: AB shall be div^{d} in the p^{ts} F, G sim^{ly} to AC.



Through D draw DHK || AB, then each of the fig^{s} FH, HB is a ▱,

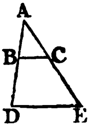
and $\therefore$ DH = FG, HK = GB : 34. 1.
but $\because$ HE is $\parallel$ KC, a side of $\triangle$ DKC,
$\therefore$ CE : ED :: KH : HD ; 2. 6.
and KH = GB, HD = FG :
$\therefore$ CE : ED :: BG : FG : 7. 5.
again, $\because$ FD is $\parallel$ GE, a side of $\triangle$ AGE,
$\therefore$ ED : DA :: GF : FA : 2. 6.
and, from above,
CE : ED :: BG : FG.

$\therefore$ *the given straight line* AB *is divided similarly to* AC. [Q. E. F.]

PROP. XI. PROB.

To find a third proportional to two given straight lines.

Let AB, AC be the two given $|^s$: it is req^d to find a third $::^1$ to them.



Place AB, AC so as to contain any $\angle$; prod. them to the p^{ts} D, E ; and make BD = AC ; join BC, and through D draw DE $\parallel$ BC : 31. 1.

CE shall be a third $::^1$ to AB, AC.

For, $\because$ BC is $\parallel$ DE, a side of $\triangle$ ADE,
$\therefore$ AB : BD :: AC : CE : 2. 6.
but BD = AC ;
$\therefore$ AB : AC :: AC : CE. 7. 5.

$\therefore$ *to the two given straight lines* AB, AC, *is found a third proportional* CE. [Q. E. F.]

PROP. XII. PROB.

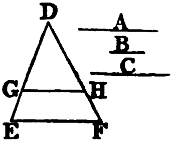
To find a fourth proportional to three given straight lines.

Let A, B, C, be the three given |^s : it is req^d to find a fourth $::^1$ to A, B, C.

Take two |^s DE, DF, containing any $\angle$ EDF :

3. 1. on them make DG = A, GE = B, and DH = C; join GH, and through E draw

31. 1. EF || GH : HF shall be a fourth $::^1$ to A, B, C.



For,

∵ GH is || EF, a side of △ DEF,

2. 6. ∴ DG : GE :: DH : HF :

but DG = A, GE = B, DH = C ;

7. 5. ∴ A : B :: C : HF.

And ∴ *to the three given straight lines* A, B, C, *is found a fourth proportional* HF [Q. E. F.]

PROP XIII. PROB.

To find a mean proportional between two given straight lines.

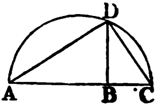
Let AB, BC be the two given |^s: it is req^d to find a mean $::^1$ between them.

Place AB, BC in one | ; on AC desc. the $\frac{1}{2}$ ⊙ ADC; and from the p^t B draw BD at

11. 1. r^t $\angle^s$ to AC : BD shall be a mean $::^1$ between AB and BC.

Join AD, DC : then the

31. 3. $\angle$ ADC in a $\frac{1}{2}$ ⊙ is a r^t $\angle$:



and ∵ in the $r^t \angle^d \triangle$ ADC, there is drawn from the $r^t \angle$ a $\perp$ BD to the base,

∴ BD is a mean $::^l$ between AB, BC, the seg^{ts} of the base, Cor. 8. 6.

∴ *between the two given straight lines* AB, BC, *a mean proportional* DB *is found.*

[Q. E. F.]

PROP. XIV. THEOR.

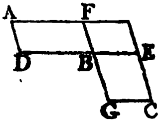
Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms that have one angle of the one equal to one angle of the other and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal $▱^s$, and let them also have the $\angle^s$ at B equal: the sides about these equal $\angle^s$ shall be reciprocally $::^l$, viz.

DB : BE :: GB : BF.

Let the sides DB, BE be placed in the same |, whence also* FB, BG will be in one |; and complete the ▱ FE: then, 14. 1.

* By hyp. DBF = $\angle$ GBE; Hyp.
add to each $\angle$ FBE;
then, $\angle^s$ (DBF + FBE) = $\angle$ (GBE+FBE); Ax. 2.
but $\angle^s$ (DBF + FBE) = two $r^t \angle^s$; 13. 1.
∴ $\angle^s$ (GBE + FBE) = two $r^t \angle^s$; Ax. 1.
and ∴ FB, BG are in the same |. 14. 1.



$\because \square AB = \square BC$,

7.5. and that FE is another $\square$,

1.6. $\therefore AB : FE :: BC : FE$:

but $AB : FE ::$ base $DB : BE$,

and $BC : FE ::$ base $GB : BF$:

11.5. $\therefore DB : BE :: GB : BF$:

$\therefore$ the sides of the $\square^s$ AB, BC about their equal $\angle^s$ are reciprocally $::^l$.

Next let the sides about the equal $\angle^s$ be reciprocally $::^l$, viz.

$DB : BE :: GB : BF$:

then shall $\square AB = \square BC$.

For,

1.6. $\because DB : BE :: GB : BF$;

and $DB : BE :: \square AB : \square FE$,

11.5. and $GB : BF :: \square BC : \square FE$:

$\therefore AB : FE :: BC : FE$;

9.5. *i.e.* AB, BC have each the same r^o to FE;

and $\therefore \square AB = \square BC$,

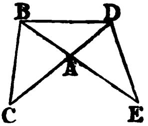
$\therefore$ *equal parallelograms, &c.* [Q. E. D.]

PROP. XV. THEOR.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal $\triangle^s$, and also let $\angle$ BAC $=$ $\angle$ DAE: the sides about the equal $\angle^s$ of the $\triangle^s$ shall be reciprocally $::^1$, viz.

CA : AD :: EA : AB.



Let the $\triangle^s$ be placed so that their sides CA, AD be in one $|$; whence also, EA, AB will be* in one $|$; and join BD: then, 14. 1.

$\because$ $\triangle$ ABC $=$ $\triangle$ ADE,

and that ABD is another $\triangle$,

$\therefore$ $\triangle$ ABC : $\triangle$ ABD :: $\triangle$ ADE : $\triangle$ ABD, 7. 5.

but $\triangle$ ABC : $\triangle$ ABD :: base AC : AD, 1. 6.

and $\triangle$ ADE : $\triangle$ ABD :: base AE : AB: 1. 6.

$\therefore$ AC : AD :: AE : AB: 11. 5.

$\therefore$ the sides of the $\triangle^s$ ABC, ADE about the equal $\angle^s$ are reciprocally $::^1$.

Next, let the sides of the $\triangle^s$ ABC, ADE about the equal $\angle^s$ be reciprocally proportional, viz.

CA : AD :: EA : AB:

then shall $\triangle$ ABC $=$ $\triangle$ ADE.

Join BD as before, then,

$\because$ CA : AD :: EA : AB;

and CA : AD :: $\triangle$ ABC : $\triangle$ BAD; 1. 6.

and EA : AB :: $\triangle$ EAD : $\triangle$ BAD: 1. 6.

$\therefore$ $\triangle$ ABC : $\triangle$ BAD :: $\triangle$ EAD : $\triangle$ BAD, 11. 5.

i.e. $\triangle^s$ ABC, AED have each the same r° to BAD,

and $\therefore$ $\triangle$ ABC $=$ $\triangle$ AED. 9. 5.

$\therefore$ *equal triangles, &c.* [Q. E. D.]

* See the note to the last proposition.

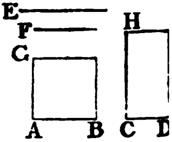
PROP. XVI. THEOR.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four |s AB, CD, E, F, be ::ls, viz.

AB : CD :: E : F:

then shall the rect. AB. F = the rect. CD. E.



From the pts A, C draw AG,
11.1. CH at rt $\angle$s to AB, CD,
3.1. making AG = F, CH = E; and
31.1. complete ▱s BG, DH: then,

∵ AB : CD :: E : F,

and that E = CH, F = AG,

7.5. ∴ AB : CD :: CH : AG:

∴ the sides of the ▱s BG, DH, about the equal $\angle$s are reciprocally ::l;

14.6. and ∴ ▱ BG = ▱ DH:

but,

BG is contained by the |s AB, AG, of wh AG = F

DH is contained by the |s CD, CH, of wh CH = E

∴ the rect. AB. F = the rect. CD. E.

And if the rect. contained by the |s AB, F be = that contained by CD, E; these |s are ::ls, viz. AB : CD :: E : F:

For the same constrn being made,

∵ AG = F, and CH = E,

∴ AB. F = AB. AG = ▱ BG,

and CD. E = CD. CH = ▱ DH:

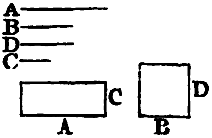
but, by hyp., AB. F = CD. E :
∴ ▱ BG = ▱ DH ; Ax. 1.
and these ▱s are equiang^r :
but the sides about the equal ∠s of equal ▱s
are reciprocally ::^{l} : 14. 6.
∴ AB : CD :: CH : AG :
but CH = E, AG = F ;
∴ AB : CD :: E : F. 7. 5.

∴ *if four &c.* [Q. E. D.]

PROP. XVII. THEOR.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: and if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Let three |s A, B, C, be ::^{ls}, viz. A : B :: B : C : then shall the rect. A. C = B^2.



Take D = B : then,
∵ A : B :: B : C,
and that D = B,
∴ A : B :: D : C ; 7. 5.
∴ the rect. A. C = B. D : 16. 6.
but ∵ B = D,
∴ B. D = B^2 :
∴ the rect. A. C = B^2.

Next, let the rect. contained by A, C = the sq. of B : then shall A, B, C be ::^{ls}, or A : B :: B : C.

For, the same constr[n] being made,

∵ B = D,

∴ the rect. B. D = B^2:

but the rect. A. C = B^2:

∴ the rect. A. C = the rect. B. D;

16. 6. ∴ the four |s A, B, D, C are ::ls,

or A : B :: D : C;

but D = B,

∴ A : B :: B : C.

∴ *if three straight lines, &c.* [Q. E. D.]

PROP. XVIII. PROB.

Upon a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let AB be the given |, and CDEF the given rect[l] fig. of four sides: it is req[d] to desc. on AB a rect[l] fig. sim[r] and sim[ly] situated to CDEF.

Join DF, and at the p[ts] A, B, in the | AB,

23. 1. make ∠ BAG = DCF, ∠ ABG = CDF;

32. 1. & then will the rem[g] ∠ AGB = the rem[g] ∠ CFD;

Ax. 3. and ∴ △ FCD be equiang[r] to △ GAB:

again, at the p[ts] G, B, in the | GB,

23. 1. make ∠ BGH = DFE, ∠ GBH = FDE;

then will the rem[g] ∠ FED = the rem[g] ∠ GHB

and ∴ △ FDE be equiang[r] to △ GBH:

and, ∵ ∠ AGB = CFD, ∠ BGH = DFE,

Ax. 2. ∴ the whole ∠ AGH = the whole ∠ CFE;

For the same reason,

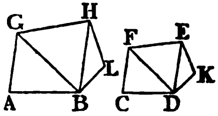
$\angle$ ABH = $\angle$ CDE:

also, $\angle$ BAG = DCF,

and $\angle$ GHB = FED:

$\therefore$ the rectl fig. ABHG

is equiangr to CDEF.



Constr.

Likewise, the sides of these figs about the equal $\angle$s are $::^{ls}$: for

$\because$ $\triangle$ GAB is equiangr to $\triangle$ FCD,

$\therefore$ BA : AG :: DC : CF, 4. 6.

AG : GB :: CF : FD;

also, $\because$ $\triangle$ BGH is equiangr to $\triangle$ DFE,

$\therefore$ GB : GH :: FD : FE;

but AG : GB :: CF : FD;

$\therefore$ *ex æq.* AG : GH :: CF : FE: 22. 5.

and in the same manner it may be proved

that AB : BH :: CD : DE;

and GH : HB :: FE : ED. 4. 6.

Hence, the rectl figs ABHG, CDEF are equiangr and the sides about their equal $\angle$s are $::^{ls}$;

and $\therefore$ the figs are simr to each other. Def.1.6

Next, let it be reqd to desc. on a given | AB, a rectl fig. simr, and simly situated, to the rectl fig. CDKEF of five sides.

Join DE, and by the preceding case, desc. on the given | AB the rectl fig. ABHG simr and simly situated to the quadrilateral fig. CDEF: and at the pts B, H, in the | BH, make

$\angle$ HBL = EDK, $\angle$ BHL = DEK:

then will the remg $\angle$ at K = the remg $\angle$ at L: 32. 1. & Ax. 3.

and ∵ the fig. ABHG is sim[r] to CDEF,

Def. 1. 6. ∴ ∠ GHB = FED ;

Constr. and ∠ BHL = DEK ;

∴ the whole ∠ GHL = the whole FEK :

for the same reason, ∠ ABL = CDK :

∴ the five-sided fig[s] AGHLB, CFEKD are equiang[l]

and ∵ fig. AGHB is sim[r] to CFED,

Def. 1. 6. ∴ GH : HB :: FE : ED ;

4. 6. but HB : HL :: ED : EK ;

22. 5. ∴ *ex æq.* GH : HL :: FE : EK :

for the same reason,

AB : BL :: CD : DK :

and ∵ △ BLH is equiang[r] to △ DKE,

4 6. ∴ BL : LH :: DK : KE.

Hence, the five-sided fig[s] AGHLB, CFEKD a[re] equiang[r], and their sides about the equal ∠[s] are :: [l]
and ∴ the fig[s] are sim[r] to each other.

In the same manner a rect[l] fig. of six sides m[ay] be desc[d] on a given | sim[r] to one given, and so on

[Q. E. F.]

PROP. XIX. THEOR.

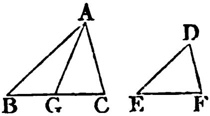
Similar triangles are to one another in the dup[licate] ratio of their homologous sides.

Let ABC, DEF be two sim[r] △[s], i[n] ∠ B = ∠ E, and AB : BC :: DE : EF ; [so]

Def. 12. 5. the side BC may be homol. to EF : △ AB[C] have to DEF the dupl. r[o] of that w[h] BC has

Take BG a third $::^l$ to BC, EF, so that 11. 6.
BC : EF :: EF : BG; and join GA: then,

$\because$ AB : BC :: DE : EF,
$\therefore$ AB : DE :: BC : EF: 16. 5.
but BC : EF :: EF : BG; Constr.
$\therefore$ AB : DE :: EF : BG: 11. 5.
$\therefore$ in the $\triangle^s$ ABG, DEF,



the sides about the equal $\angle^s$ are reciprocally $::^l$;
and $\therefore$ $\triangle$ ABG $=$ $\triangle$ DEF. 15. 6.

And, $\because$ BC : EF :: EF : BG,
and that, if three $|^s$ be $::^l$, the 1st is said to have to the 3rd the dupl. r° of that wh it has to the 2nd; Def. 10. 5.
$\therefore$ BC has to BG the dupl. r° of that wh BC has to EF:
but $\triangle$ ABC : $\triangle$ ABG :: BC : BG: 1. 6.
$\therefore$ $\triangle$ ABC has to $\triangle$ ABG the dupl. r° of that wh BC has to EF:
but $\triangle$ ABG $=$ $\triangle$ DEF;
$\therefore$ also $\triangle$ ABC has to $\triangle$ DEF the dupl. r° of that wh BC has to EF.

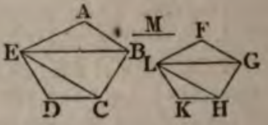
$\therefore$ *similar triangles, &c.* [Q. E. D.]

Cor.—From this it is manifest, that if three $|^s$ be $::^{ls}$, as the 1st is to the 3rd, so is any $\triangle$ on the 1st to a simr and simly descd $\triangle$ on the 2nd

PROP. XX. THEOR.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be sim^r polygons, and let AB be the side homol. to FG: the polygons may be div^d into the same n^o of sim^r $\triangle^s$, whereof each shall have to each the same r^o w^h the polygons have; and the polygon ABCDE shall have to FGHKL the dupl. r^o of that w^h AB has to FG.



Join BE, EC, GL, LH: then,

$\because$ the polygon ABCDE is sim^r to FGHKL

Def.1.6. $\therefore \angle BAE = \angle GFL$,

Def 1.6. and also $BA : AE :: GF : FL$,

i.e. one $\angle$ of $\triangle ABE$ = one $\angle$ of $\triangle FGL$,

and also the sides about these equal $\angle^s$ are $::^{ls}$;

6. 6. $\therefore \triangle ABE$ is equiang^r to $\triangle FGL$,

4 6. and $\therefore$ the $\triangle^s$ are sim^r to one another:

and, $\because \triangle ABE$ is sim^r to $\triangle FGL$,

Def.1.6. $\therefore \angle ABE = \angle FGL$;

also $\because$ the polygons are sim^r,

$\therefore$ the whole $\angle ABC$ = the whole $\angle FGH$;

Ax. 3. $\therefore$ the rem^g $\angle EBC$ = the rem^g $\angle LGH$:

and $\because \triangle ABE$ is sim^r to $\triangle FGL$,

Def.1.6 $\therefore EB : BA :: LG : GF$;

and also $\because$ the polygons are sim^r,

Def.1.6. $\therefore AB : BC :: FG : GH$;

22. 5. $\therefore$ *ex æq.* $EB : BC :: LG : GH$,

6. 6. *i.e.* the sides about the equal $\angle^s$ EBC, LGH are $::^{ls}$;

4. 6. and $\therefore \triangle EBC$ is equiang^r and sim^r to $\triangle LGH$:

for the same reason,

△ ECD is equiang^r and sim^r to △ LHK:
and ∴ the sim^r polygons ABCDE, FGHKL are div^d into the same n° of sim^r △^s.

Also, these △^s shall have, each to each, the same r° w^h the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE shall have to FGHKL the dupl. r° of that w^h the side AB has to the homol. side FG.

For, ∵ △ ABE is sim^r to △ FGL

∴ ABE has to FGL the dupl. r° of that w^h the side BE has to the side GL: 19. 6.

for the same reason,

△ BEC has to GLH the dupl. r° of that w^h BE has to GL:

and ∴ △ ABE : FGL :: BEC : GLH. 11. 5.

Again,

∵ △ EBC is sim^r to △ LGH,

∴ EBC has to LGH the dupl. r° of that w^h the side EC has to the side LH:

for the same reason,

△ ECD has to LHK the dupl. r° of that w^h EC has to LH:

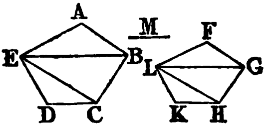
∴ △ EBC : LGH :: ECD : LHK: 11. 5.

but it has also been proved that

△ EBC : LGH :: ABE : FGL;

∴ △ ABE : FGL :: EBC : LGH
:: ECD : LHK:

and ∴ as one antecedent is to its consequent, 12. 5.
so are all the antecedents to all the consequents; *i.e.*
polygon ABCDE : FGHKL :: △ ABE : FGL;



19. 6. but $\triangle$ ABE has to FGL the dupl. r^{o} of that w^{h} the side AB has to the homol. side FG;
and ∴ polygon ABCDE has to FGHKL the dupl. r^{o} of that w^{h} AB has to the homol. side FG.

∴ *similar polygons, &c.* [Q. E. D.]

Cor. 1—In like manner it may be proved that sim^{r} four-sided fig^{s}, or of any n^{o} of sides, are one to another in the dupl. r^{o} of their homol. sides:
19. 6. and it has already been proved in the case of $\triangle^{s}$:
∴ universally, sim^{r} rect^{l} fig^{s} are to one another in the dupl. r^{o} of their homol. sides.

Cor. 2.—And if to AB, FG, two of the homol.
11. 6. sides, a third $::^{l}$ M be taken,
Def. 10. 5. AB has to M the dupl. r^{o} of that w^{h} AB has to FG:
but the four-sided fig. or polygon on AB has to the four-sided fig. or polygon on FG also the dupl. r^{o}
Cor. 1. of that w^{h} AB has to FG;
11. 5 ∴ AB : M :: the fig. on AB : fig. on FG:
Cor. 19. 6. and this was also proved in the case of $\triangle^{s}$:
∴ universally, if three $|^{s}$ be $::^{ls}$,
as the 1st is to the 3rd so is any rect^{l} fig. on the 1st to a sim^{r} and sim^{ly} desc^{d} rect^{l} fig. on the 2nd.

PROP. XXI. THEOR.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rect[l] fig[s] A, B be sim[r] to the rect[l] fig. C: the fig. A shall be sim[r] to the fig. B.

For, ∵ A is sim[r] to C,

∴ they are equiang[r],

and also their sides about the equal ∠[s] are ∷[l]: Def.1.6.

again,

∵ B is sim[r] to C,

∴ they are equiang[r],

and their sides about the equal ∠[s] are ∷[ls]: Def.1.6.

∴ the fig[s] A, B are each of them equiang[r] to C, and the sides about the equal ∠[s] of each of them and of C are ∷[ls]:

∴ the rect[l] fig[s] A and B are equiang[r], Ax. 1. 1.

and their sides about their equal ∠[s] are ∷[ls]; 11. 5.

∴ A is sim[r] to B. Def.1 6.

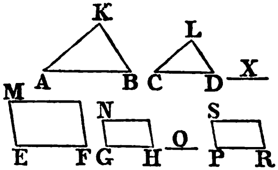
∴ *rectilineal figures, &c.* [Q. E. D.]

PROP. XXII. THEOR.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals: and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four $|^{s}$ AB, CD, EF, GH be $::^{ls}$,
viz. AB : CD :: EF : GH ;
and on AB, CD let the sim^{r} rect^{l} fig^{s} KAB, LCD be sim^{ly} desc^{d} : and on EF, GH the sim^{r} rect^{l} fig^{s} MF, NH, in like manner:
then shall the rect^{l} fig. KAB : LCD :: MF : NH

11. 6. To AB, CD take a third $::^{l}$ X,
and to EF, GH a third $::^{l}$ O:
then, $\because$ AB : CD :: EF : GH,
11. 5. $\therefore$ CD : X :: GH : O;
22. 5. $\therefore$ *ex æq.* AB : X :: EF : O:
Cor. 2. but AB : X :: fig. KAB : fig. LCD;
20. 6. and EF : O :: fig. MF : fig. NH:
11. 5. $\therefore$ KAB : LCD :: MF : NH.



Next,
let the rect^{l} fig. KAB : LCD :: MF : NH :
then shall | AB : CD :: EF : GH.

12. 6. Make AB : CD :: EF : PR,
18. 6. and on PR desc. the rect^{l} fig. SR sim^{r} and sim^{ly} situated to either of the fig^{s} MF, NH :
then, $\because$ AB : CD :: EF : PR,
and that on AB, CD are desc^{d} the sim^{r} and sim^{ly} situated rect^{l} fig^{s} KAB, LCD, and on EF, PR, in like manner, the sim^{r} rect^{l} fig^{s} MF, SR ;
$\therefore$ KAB : LCD :: MF : SR :

but, by hyp. KAB : LCD :: MF : NH ;
i.e. the rect^l MF has the same r^o to each of the two NH, SR,
and ∴ NH = SR: 9. 5.
and these fig^s are sim^r, and sim^{ly} situated:
∴ GH = PR:
and ∵ AB : CD :: EF : PR,
and that PR = GH,
∴ AB : CD :: EF : GH. 7. 5.

∴ *if four straight lines, &c.* [Q. E. D.]

PROP. XXIII. THEOR.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

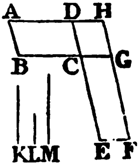
Let AC, CF be equiang^r ▱^{s}, having the ∠ BCD = ECG: the r^o of ▱ AC to ▱ CF shall be the same with the r^o w^h is compounded of the r^{os} of their sides.

Let BC, CG be placed in a | ; whence also DC, CE* will be in a | ; complete ▱ DG ; and taking any | K, 14. 1.
make K : L :: BC : CG, 12. 6.
and L : M :: DC : CE :
then the r^{os} of K to L and L to M are the same with the r^{os} of the sides, viz. of BC to CG and DC to CE:

* See the note to Prop. 14. 6.

Def. A. 5. but the r^o of K to M is that which is said to be compounded of the r^{os} of K to L and L to M.

∴ K has to M the r^o compounded of the r^{os} of the sides:



and

1. 6. ∵ BC : CG :: ▱AC : ▱CH;

but

BC : CG :: K : L;

11. 5 ∴ K : L :: ▱AC : ▱CH;

again, ∵ DC : CE :: ▱CH : ▱CF;

but DC : CE :: L : M;

11. 5. ∴ L : M :: ▱CH : ▱CF;

hence, it having been proved,

that K : L :: ▱AC : ▱CH,

and L : M :: ▱CH : ▱CF;

22. 5. ∴ *ex æq.* K : M :: ▱AC : ▱CF:

but K has to M the r^o w^h is compounded of the r^{os} of the sides:

∴ also ▱AC has to ▱CF the r^o w^h is compounded of the r^{os} of the sides.

∴ *equiangular parallelograms, &c.*

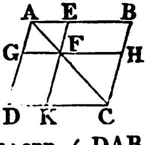
[Q. E. D.]

PROP. XXIV. THEOR.

Parallelograms about the diameter of any parallelogram, are similar to the whole, and to one another.

Let ABCD be a ▱ of w^h the $diam^r$ is AC;

and EG, HK ▱s about the diamr: these ▱s shall be simr to the whole ▱ ABCD, and to one another.



For, ∵ DC is ∥ GF,

∴ ∠ ADC = ∠ AGF; 29. 1.

and ∵ BC is ∥ EF,

∴ ∠ ABC = ∠ AEF:

also

∵ the ∠s BCD, EFG are each = the opp. ∠ DAB, 34. 1.

∴ ∠ BCD = ∠ EFG;

and ∴ ▱ ABCD is equiangr to ▱ AEFG:

again, ∵ ∠ ABC = ∠ AEF,

and ∠ BAC is com. to the two △s BAC, EAF,

∴ these △s are equiangr to one another;

and ∴ AB : BC :: AE : EF: 4. 6.

and the opp. sides of ▱s are = one another; 34. 1.

whence AB : AD :: AE : AG; 7. 5.

and DC : CB :: GF : FE,

and also CD : DA :: FG : GA:

i. e. the sides of the ▱s ABCD, AEFG about the equal ∠s are ::ls;

and ∴ these ▱s are simr to each other. Def. 1 6

for the same reason,

▱ ABCD is simr to ▱ FHCK:

and ∴ each of the ▱s GE, KH is simr to DB:

but rectl figs wh are simr to the same rectl fig. are also simr to each other; 21. 6.

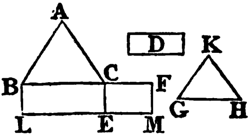
and ∴ ▱ GE is simr to ▱ KH.

∴ *parallelograms, &c.* [Q. E. D.]

PROP. XXV. PROB.

To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

Let ABC be the given rect^l fig. to w^h the fig. to be desc^d is req^d to be sim^r, and D that to w^h it must be equal: it is req^d to desc. a rect^l fig. sim^r to ABC and = D.



Cor. 45. 1. On the | BC desc. the ▱ BE = the fig. ABC;
Cor. 45. 1. also on CE desc. the ▱ CM = D, and having $\angle$ FCE = $\angle$ CBL:
29. 1. & 14. 1. then BC, CF will be in one |, as also LE and EM*:

Constr. * By constr^n, $\angle$ FCE = $\angle$ CBL;
add $\angle$ ECB to each;
Ax. 2. ∴ $\angle^s$ (FCE + ECB) = $\angle^s$ (ECB + CBL)
29. 1. but $\angle^s$ (ECB + CBL) = two r^t $\angle^s$;
Ax. 1. ∴ $\angle^s$ (FCE + ECB) = two r^t $\angle^s$;
14. 1. and ∴ BC, CF are in the same |.
Again, ∵ $\angle$ LBC = $\angle$ FCE,
34. 1. and that $\angle$ LBC = the opp. $\angle$ LEC;
Ax. 1. ∴ $\angle$ LEC = $\angle$ ECF;
add $\angle$ CEM to each;
then $\angle^s$ (LEC + CEM) = $\angle^s$ (ECF + CEM):
29. 1. but $\angle^s$ (ECF + CEM) = two r^t $\angle^s$;
Ax. 1 ∴ $\angle^s$ (LEC + CEM) = two r^t $\angle^s$;
14. 1. and ∴ LE, EM are in the same |.

between BC and CF find a mean $::^{l}$ GH, and 13. 6.
on GH desc. the rect^{l} fig. KGH sim^{r} and sim^{ly}
situated to the fig. ABC. 18. 6.

Then, ∵ BC : GH :: GH : CF,
and that, if three $|^{s}$ be $::^{ls}$,
as the 1st is to the 3rd, so is the fig. on the 1st Cor. 2.
to the sim^{r} and sim^{ly} desc^{d} fig. on the 2nd; 20. 6.
∴ BC : CF :: fig. ABC : KGH :
but BC : CF :: ▱ BE : EF; 1. 6.
∴ fig. ABC : KGH :: ▱ BE : EF : 11. 5.
and the rect^{l} fig. ABC = ▱ BE; Constr.
∴ the rect^{l} fig. KGH = ▱ EF : 14. 5.
but EF = the fig. D; Constr
∴ also KGH = D,
and it is sim^{r} to ABC.

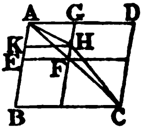
∴ *the rectilineal figure* KGH *has been described similar to the figure* ABC *and equal to* D.

[Q. E. F.]

PROP. XXVI. THEOR.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let the ▱^{s} ABCD, AEFG be sim^{r} and sim^{ly} situated, and have the ∠ DAB com. to both: the ▱^{s} shall be about the same diam^{r}.



For, if not, let, if possible, ▱ BD have its diam^r AHC in a different | from AF, the diam^r of ▱ EG, and let GF meet AHC in H; and through H

34. 1. draw HK ∥ AD or BC: then,

$\because$ ▱^s ABCD, AKHG are about the same diam^r,

24. 6. $\therefore$ they are sim^r to each other;

Def.1.6. and $\therefore$ DA : AB :: GA : AK:

Hyp. but $\because$ ABCD, AEFG are sim^r ▱^s,

$\therefore$ DA : AB :: GA : AE;

11. 5. $\therefore$ GA : AE :: GA : AK,

i.e. GA has the same r^o to each of the |^s AE, AK

9. 5. and $\therefore$ AK = AE,

or the less = the greater,

w^h is impossible;

$\therefore$ ABCD, AKHG are not about the same diam^r,

$\therefore$ ABCD, AEFG must be about the same diam^r,

$\therefore$ *if two similar, &c.* [Q. E. D.]

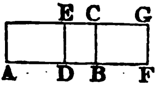
To understand the three following propositions* more easily, it is to be observed, that

1. A ▱ is said to be applied to a |, when it is desc^d on it as one of its sides; *ex. gr.* the ▱ AC is said to be applied to the | AB.

2. But a ▱ AE is said to be applied to a | AB, deficient by a ▱, when AD, the base of AE, is < AB, and $\therefore$ AE is < the ▱ AC desc^d

* These three propositions are seldom read in the University.

on AB in the same ∠, and between the same $\parallel^{s}$, by the ▱ DC; and DC is ∴ called the defect of AE.



3. And a ▱ AG is said to be applied to a | AB, exceeding by a ▱, when AF the base of AG is > AB, and ∴ AG exceeds AC, the ▱ desc^{d} on AB in the same ∠, and between the same $\parallel^{s}$, by the ▱ BG.

PROP. XXVII. THEOR.

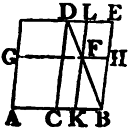
Of all parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half the line; that which is applied to the half, and is similar to its defect, is the greatest.

Let AB be a | div^{d} into two equal parts in C; and let the ▱ AD be applied to the half AC, w^{h} ∴ is deficient from the ▱ on the whole | AB by the ▱ CE upon the other half CB: of all the ▱^{s} applied to any other part of AB, and deficient by ▱^{s} that are sim^{r} and sim^{ly} situated to CE, AD shall be the greatest.

Let AF be any ▱ applied to AK, any other part of AB than the half, so as to be deficient from

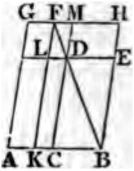
the ▱ on the whole | AB by the ▱ KH sim^r and sim^{ly} situated to CE: AD shall be > AF.

First, let AK, the base of AF, be > AC, the half of AB:



Hyp. ∵ ▱ CE is sim^r to ▱ KH,
26. 6. ∴ they are about the same diam^r; draw this diam^r DB, and complete the scheme: then,
43. 1. ▱ CF = ▱ FE;
add KH to both:
∴ the whole CH = the whole KE:
but ∵ the base AC = the base CB,
36. 1. ∴ CH = CG;
Ax. 1. ∴ ▱ CG = KE:
add CF to both;
Ax. 2. then the whole AF = the gnomon CHL;
∴ CE, or the ▱ AD, is > the ▱ AF.

Next, let AK be < AC:
then, the same constr^n being made,



∵ BC = CA,
4. 1. ∴ HM = MG;
6. 1. ∴ ▱ DH = ▱ DG;
and ∴ DH > LG:
43. 1. but DH = DK;
∴ DK > LG:
add AL to both:
then the whole AD > the whole AF.

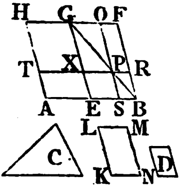
∴ *of all parallelograms applied, &c.*

[Q. E. D.]

PROP. XXVIII. PROB.

To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram: but the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram 27. 6.
applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram.

Let AB be the given |, and C the rect[l] fig. to w[h] the ▱ to be applied is req[d] to be equal, w[h] fig. must not be > the ▱ applied to the half of the |, having its defect from that on the whole | sim[r] to the defect of that w[h] is to be applied; and let D be the ▱ to w[h] this defect is req[d] to be sim[r]: it is req[d] to apply to the | AB a ▱ w[h] shall be = the fig. C, and be deficient from the ▱ on the whole | by a ▱ sim[r] to D.



Div. AB into two equal parts in the p[t] E, on EB 10. 1.
desc. ▱ EBFG sim[r] and sim[ly] situated to D, and complete ▱ AG, w[h], by the determination, must either be = C, or > it. 18. 6.

If AG = C, then what was req[d] is already done: for, on the | AB is applied the ▱ AG = the fig. C, and deficient by the ▱ EF sim[r] to D.

But if AG be $\not\subset$ C, it is $>$ it:
36. 1. and EF is $=$ AG;
$\therefore$ also EF is $>$ C.

25 6. Make the $\square$ KLMN $=$ the excess of EF above C, and sim^r and sim^ly situated to D: then
Constr. $\because$ D is sim^r to EF,
21. 6. $\therefore$ also KM is sim^r to EF:

let KL be the homol. side to EG, and LM to GF:

$\because$ EF $=$ C $+$ KM,
$\therefore$ EF $>$ KM;
$\therefore$ | EG $>$ KL,
and GF $>$ LM:
3. 1. make GX $=$ LK, GO $=$ LM,
31. 1. and complete $\square$ GOP:

then XO is $=$ and sim^r to KM:
but KM is sim^r to EF;
$\therefore$ also XO is sim^r to EF;
26. 6. and $\therefore$ XO and EF are about the same diam^r:
let GPB be their diam^r, and complete the scheme.

Then, $\because$ EF $=$ C $+$ KM,
of w^h, the part XO $=$ the part KM,
Ax. 3. $\therefore$ the rem^r ERO $=$ the rem^r C;

43. 1. Again, $\because$ OR $=$ XS,
$\therefore$ the whole OB $=$ the whole XB:
but $\because$ the base AE $=$ the base EB,
36. 1. $\therefore$ XB $=$ TE;
Ax. 1. $\therefore$ also TE $=$ OB:
add XS to each,
then, the whole TS $=$ the gnomon ERO

but, from above, $ERO = C$;
$\therefore$ also $TS = C$.

$\therefore$ *the parallelogram* TS, *which is equal to the given rectilineal figure* C, *is applied to the given straight line* AB *deficient by the parallelogram* SR, 24. 6.
similar to the given one D, *since* SR *is similar to* EF. [Q. E. F.]

PROP. XXIX. THEOR.

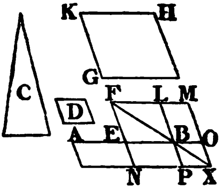
To a given straight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogrom similar to another given.

Let AB be the given |, C the given rect^{l} fig. to w^{h} the ▱ to be applied is req^{d} to be equal, and D the ▱ to w^{h} the excess of the one to be applied above that on the given | is req^{d} to be sim^{r}: it is req^{d} to apply to the given | AB a ▱ w^{h} shall be = the fig. C, exceeding by a ▱ sim^{r} to D.

Bis^{t} AB in the p^{t} E; on EB desc. the ▱ EL 10. 1.
sim^{r} and sim^{ly} situated to D; and make the 18. 6.
▱ $GH = EL + C$, and sim^{r} and sim^{ly} situated 25. 6.
to D; whence also GH is sim^{r} to EL: let KH be 21. 6.
the side homol. to FL, and KG to FE: then,

$\because$ ▱ GH is $>$ EL,
$\therefore$ side KH is $>$ FL,
and $KG > FE$:

Prod. FL and FE, making FLM = KH
FEN = KG; and complete the ▱ MN;
then MN is equal and sim^r to GH:
but GH is sim^r to EL;
∴ MN is sim^r to EL;
∴ EL and MN are
26. 6. about the same diam^r:
draw this diam^r FX, and
complete the scheme.



Then,
∵ GH = EL + C,
and that GH = MN,
∴ MN = EL + C;
take away the com. part EL;
then the rem^r NOL = the rem^r C.

Again ∵ AE = EB,
36. 1. ∴ ▱ AN = ▱ NB,
43. 1. = ▱ BM:
add NO to each;
then ▱ AX = the gnomon NOL
but, from above, NOL = C;
∴ also AX = C.

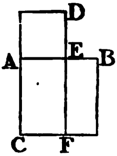
∴ *to the straight line* AB *is applied the parallelogram* AX *equal the given rectilineal figure* C, *exceeding by the parallelogram* PO, *which is similar*
24. 6. *to* D, *since* PO *is similar to* EL. [Q. E. F.]

PROP. XXX. THEOR.

To cut a given straight line in extreme and mean ratio.

Let AB be the given |: it is req[d] to cut it in extreme and mean r°.

On AB desc. the sq. BC, and to AC apply the ▱ CD = BC, and exceeding by the fig. AD sim[r] to BC: then, 46. 1. 29. 6.



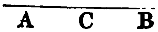
$\because$ BC is a sq.,
$\therefore$ also AD is a sq.:
and $\because$ BC = CD,
and that CE is com. to both,
$\therefore$ rem[r] BF = rem[r] AD:
and these fig[s] are equiang[r]:
$\therefore$ their sides about the equal $\angle$[s] are reciprocally $::^1$: 14. 6.
and $\therefore$ FE : ED :: AE : EB:
but FE = AC = AB, and ED = AE; 34. 1. Def. 30.
$\therefore$ AB : AE :: AE : EB:
but AB is > AE,
$\therefore$ AE is > EB. 14. 5.

$\therefore$ *the straight line* AB *is cut in extreme and mean ratio in* E. Def.3.6

[Q. E. F.]

Otherwise,

Div. AB in the p[t] C, so that the rect. contained by AB, BC may be = the sq. of AC: then, 11. 2.



$\because$ the rect. AB. BC = AC^2,
$\therefore$ AB : AC :: AC : BC. 17. 6.

$\therefore$ AB *is cut in extreme and mean ratio in* C. Def.3.6.

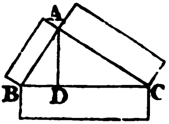
[Q.E.F.]

PROP. XXXI. THEOR.

In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly-described figures, upon the sides containing the right angle.

Let ABC be a r^t^ $\angle^d$ $\triangle$, BAC being the r^t^ $\angle$: the rect^l^ fig. desc^d^ on BC shall be $=$ the sim^r^ and sim^ly^ desc^d^ fig^s^ on BA, AC.

12. 1. Draw the $\perp$ AD: then,
$\because$ from the r^t^ $\angle$ BAC of
$\triangle$ ABC is drawn the $\perp$ AD
to the base,
$\therefore$ $\triangle^s$ ABD, ADC are sim^r^
8. 6. to the whole $\triangle$ ABC, and
to one another:



and $\because$ $\triangle$ ABC is sim^r^ to ADB,
4. 6. $\therefore$ CB : BA :: BA : BD :
and $\because$ these three $|^s$ are $::^{ls}$,
$\therefore$ as the 1^st^ is to the 3^rd^, so is the fig. on the 1^st^
Cor. 2. 20. 6. to the sim^r^ and sim^ly^ desc^d^ fig. on the 2^nd^:
$\therefore$ as CB to BD, so is the fig. on CB to the sim^r^ and sim^ly^ desc^d^ fig. on AB;

and, inv^ly^,
B. 5. BD : CB :: the fig. on AB : that on BC:
for the same reason,
DC : CB :: the fig. on AC : that on BC:
24. 5. $\therefore$ BD $+$ DC : CB :: fig^s^ on AB, AC: that on BC:
but BD $+$ DC $=$ BC;
A. 5. and $\therefore$ the fig. on BC $=$ the sim^r^ and sim^ly^ desc^d^ fig^s^ on AB, AC.

$\therefore$ *in right-angled triangles, &c.* [Q. E. D.]

PROP. XXXII. THEOR.

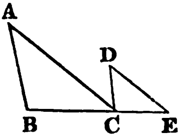
If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel to one another; the remaining sides shall be in a straight line.

Let ABC, DCE be two △s wh have the two sides BA, AC ::l to the two CD, DE, viz.

BA : AC :: CD : DE;

and let AB be ∥ DC, AC ∥ DE:

BC and CE shall be in the same |.



For,

∵ | AC meets the ∥s AB, DC,

∴ ∠ BAC = the alt. ∠ ACD; 29. 1

for the same reason,

∠ CDE = ∠ ACD;

∴ also ∠ BAC = ∠ CDE: Ax. 1

and in the △s ABC, DCE,

∵ one ∠ at A = one ∠ at D,

and the sides about these ∠s are ::ls,

viz. BA : AC :: CD : DE,

∴ △ ABC is equiangr to DCE, 6. 6.

∴ ∠ ABC = ∠ DCE: Ax. 2.

and, from above, ∠ BAC = ∠ ACD,

∴ the whole ∠ ACE = the two ∠s (ABC + BAC):

add the com. ∠ ACB; then,

∠s (ACE + ACB) = ∠s (ABC + BAC + ACB):

but ∠s (ABC + BAC + ACB) = two rt ∠s; 32. 1.

$\therefore$ also $\angle^s(ACE + ACB) =$ two $r^t \angle^s$,
i. e. at the p^t C, in the | AC, the two $|^s$ BC, CE
w^h are on the opp. sides of it,
make the $adj^t \angle^s =$ two $r^t \angle^s$;
ib. 1. and $\therefore$ BC, CE are in the same |.

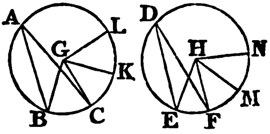
$\therefore$ *if two triangles, &c.* [Q. E. D.]

PROP. XXXIII. THEOR.

In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another: so also have the sectors.

Let ABC, DEF be equal $\odot^s$,
BGC, EHF, $\angle^s$ at their $cent^s$,
BAC, EDF, $\angle^s$ at their $\odot^{ces}$:

$$\left.\begin{array}{r} \angle BGC : \angle EHF \\ \angle BAC : \angle EDF \\ \text{sector } BGC : \text{sector } EHF \end{array}\right\} :: \text{arc } BC : \text{arc } EF$$



Take any n^o of arcs CK, KL, each $=$ BC,
and any n^o of arcs FM, MN, each $=$ EF;
and join GK, GL, HM, HN: then
$\because$ arc BC $=$ CK $=$ KL,
27. 3 $\therefore \angle BGC = \angle CGK = \angle KGL$;

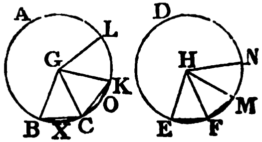
ıd ∴ whatever mult. the arc BL is of the arc BC,
the same mult. is ∠ BGL of ∠ BGC:

ır the same reason,
whatever mult. the arc EN is of the arc EF,
the same mult. is ∠ EHN of ∠ EHF:
but as the arc BL is >, = or < the arc EN,
so is ∠ BGL >, = or < ∠ EHN: 27. 3.

ıce then there are four magn^s, viz.
.e two arcs BC, EF, and the two ∠^s BGC, EHF;
and that of the arc BC, and the ∠ BGC,
have been taken any equimult^s whatever,
viz. the arc BL, and the ∠ BGL:
and also of the arc EF, and the ∠ EHF,
have been taken any equimult^s whatever,
viz. the arc EN, and the ∠ EHN;

d since it has also been proved that
as the arc BL is >, = or < the arc EN,
so is ∠ BGL >, = or < ∠ EHN;
∴ ∠ BGC : ∠ EHF :: arc BC : arc EF: Def. 5. 5.
ıt, ∵ ∠ BGC is double of ∠ BAC, 20. 3.
and ∠ EHF is double of ∠ EDF;
∴ ∠ BGC : ∠ EHF :: ∠ BAC : ∠ EDF: 15. 5.
so, ∴ ∠ BAC : ∠ EDF :: arc BC : arc EF.

Again,
arc BC to EF, so shall sector BGC be to EHF.
Join BC, CK; in the arcs BC, CK take any
^s X, O, and join BX, XC, CO, OK:
en, in the △^s GBC, GCK,
. { the sides BG, GC = those CG, GK, each to each,
and that these sides contain equal ∠^s,
{ the base BC = the base CK; 4. 1.
• and △ GBC = △ GCK:



also ∵ the arc BC = the arc CK,

3. ∴ the rem^{g} part BALC of the whole ⊙^{ce} } = { the rem^{g} part CBAK of the same ⊙^{ce},

3. and ∴ ∠ BXC = ∠ COK ;

f. 11. ∴ seg^{t} BXC is sim^{r} to seg^{t} COK ;

and they are on equal |^{s} BC, CK :

3. but sim^{r} seg^{ts} of ⊙^{s} on equal |^{s} are themselves equal;

∴ seg^{t} BXC = seg^{t} COK :

and it has been proved, that

△ BGC = △ CGK ;

∴ the sector BGC = the sector CGK :

for the same reason,

the sector KGL = each of the sectors BGC, CGK

and in the same manner it may be proved, that

the sector EHF = FHM = MHN :

∴ whatever mult. the arc BL is of the arc B(

the same mult. is the sector BGL of the sector BG(

and, for the same reason,

whatever mult. the arc EN is of EF,

the same mult. is the sector EHN of EHF

but as the arc BL is >, = or < the arc EN,

so is the sector BGL >, = or < the sector EI

since then there are four magn^{s}, viz.

the two arcs BC, EF, and the two sectors BGC, F

and that of the arc BC and the sector B(

the arc BL and sector BGL are any equim

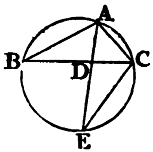
and of the arc EF and the sector EHF,
the arc EN and sector EHN are any equimult^s;
and since it has also been proved, that
as the arc BL is $>, =$ or $<$ EN,
so is the sector BGL $>, =$ or $<$ EHN;
$\therefore$ sector BGC : EHF :: arc BC : EF. Def.5.5

$\therefore$ *in equal circles, &c.* [Q. E. D.]

PROP. B. THEOR.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.

Let $\angle$ BAC of $\triangle$ ABC be bis^d by the | AD:
then shall the rect. BA. AC = rect. BD. DC + AD^2.

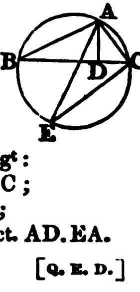


Desc. $\odot$ ABC about the $\triangle$, 5. 4.
prod. AD to the $\odot^{ce}$ in E, and
join EC: then,
$\because \angle$ BAD $= \angle$ CAE, Hyp.
and $\angle$ ABD $= \angle$ AEC, 21. 3.
for they are in the same seg^t;
$\therefore$ $\triangle$ABD is equiang^r to AEC; 32. 1.
$\therefore$ BA : AD :: EA : AC; 4. 6.
$\therefore$ the rect. BA. AC = the rect. AD. EA 16. 6.
= the rect. ED. DA + AD^2; 3. 2.
but the rect. ED. DA = the rect. BD. DC; 35. 3.
$\therefore$ the rect. BA. AC = the rect. BD. DC + AD^2.
$\therefore$ *if an angle, &c.* [Q. E. D.]

PROP. C. THEOR.

If from any angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

From ∠ BAC of △ ABC let AD be drawn ⊥ to the base BC: then shall the rect. BA.AC = the rect. contained by AD and the diam[r] of the ⊙ desc[d] about the △.



5. 4. Desc. the ⊙ ACB about the △, draw its diam[r] AE, and join EC: then,

31. 3. ∵ r[t] ∠ BDA = ∠ ECA in a $\frac{1}{2}$ ⊙,

12. 3. and ∠ ABD = ∠ AEC,

for they are in the same seg[t]:

4. 6. ∴ △ ABD is equiang[r] to AEC;

∴ BA : AD :: EA : AC;

16. 6. and ∴ the rect. BA. AC = the rect. AD. EA.

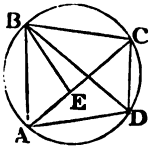
∴ *if from an angle, &c.* [Q. E. D.]

PROP. D. THEOR.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let ABCD be any quadrilat[l] fig. insc[d] in a ⊙;

and join AC, BD: the rect. contained by AC, BD shall be = the two rect^s^ contained by AB, CD, and by AD, BC.*



Make ∠ ABE = ∠ DBC:
then, adding the com. ∠ EBD, 23. 1
∠ ABD = ∠ EBC:
and ∠ BDA = ∠ BCE,
for they are in the same seg^t^; 21. 3.

∴ △ ABD is equiang^r^ to BCE;

∴ BC : CE :: BD : DA;

and ∴ the rect. CE. BD = the rect. BC. AD: 4. 6.

again, 16. 6.

∵ ∠ ABE = DBC, and ∠ BAE = BDC,

∴ △ ABE is equiang^r^ to BCD; 21. 3.

∴ BA : AE :: BD : DC;

and ∴ the rect. AE. BD = the rect. BA. DC;

but, from above,

the rect. CE. BD = the rect. BC. AD;

∴ the whole rect. AC. BD = the rect. AB. DC,
+ the rect. BC. AD. 1. 2.

∴ *the rectangle, &c.* [Q. E. D.]

* This is a Lemma of Cl. Ptolomæus, in page 9. of his μεγάλης συντάξεως.

BOOK XI.

DEFINITIONS.

I.

A SOLID is that which hath length breadth, and thickness.

II.

That which bounds a solid is a superficies.

III.

A straight line is perpendicular, or at right angles, to a plane, when it makes right angles with every straight line in that plane which meets it.

IV.

A plane is perpendicular to a plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

V.

The inclination of a straight line to a plane, is the *acute* angle, contained by that straight line, and

another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.

VI.

The inclination of a plane to a plane is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.

VII.

Two planes are said to have the same or a like inclination to one another which two other planes have, when the said angles of inclination are equal to one another.

VIII.

Parallel planes are such as do not meet one another however far they be produced.

IX.

A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

X.

'The tenth definition is omitted.'

XI.

Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

XII.

A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it, in which point they meet.

XIII.

A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others parallelograms.

XIV.

A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

XV.

The axis of a sphere is the fixed straight line about which the semicircle revolves.

XVI.

The centre of a sphere is the same with that of the semicircle.

XVII.

The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

XVIII.

A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled; and if greater, an acute-angled cone.

XIX.

The axis of a cone is the fixed straight line about which the triangle revolves.

XX.

The base of a cone is the circle described by that side containing the right angle which revolves.

XXI.

A cylinder is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

XXII.

The axis of a cylinder is the fixed straight line about which the parallelogram revolves.

XXIII.

The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

XXIV.

Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

XXV.

A cube is a solid figure contained by six equal squares.

XXVI.

A tetrahedron is a solid figure contained by four equal and equilateral triangles.

XXVII.

An octahedron is a solid figure contained by eight equal and equilateral triangles.

XXVIII.

A dodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

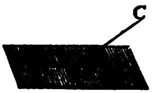
Def. A.

A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

PROP. I. THEOR.

One part of a straight line cannot be in a plane, and another part above it.

If it be possible, let AB, part of the | ABC, be in the plane, and the part BC above it: then,
∵ the | AB is in the plane,
∴ it can be prod^{d} in that plane:
let it be prod^{d} to D; and let any plane pass through the | AD, and be turned about it until it pass through the p^{t} C: then,



∵ the p^{ts} B, C are in this plane,
∴ the | BC is in it: Def.7.1.
∴ there are two |^{s} ABC, ABD in the same plane
that have a com. seg^{t} AB:
but this is impossible. Cor. 11. 1.

∴ *one part, &c.* [Q. E. D.]

PROP. II. THEOR.

Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane.

Let two |^{s} AB, CD cut one another in E; they shall be in one plane: and three |^{s} EC, CB, BE, w^{h} meet one another shall be in one plane.

Let any plane pass through the | EB, and let the plane be turned about EB, prod[d] if necessary, until it pass through the p[t] C : then,

∵ the p[ts] E, C are in this plane,

Def.7.1. ∴ the | EC is in it:

for the same reason,

the | BC is in the same plane;

and, by hyp., EB is in it:

∴ the three |[s] EC, CB, BE are in one plane:

but in the plaue in w[h] EC, EB are,

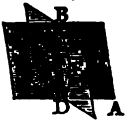
I. 11. in the same are CD, AB:

∴ AB, CD are in one plane.

∴ *two straight lines, &c.* [Q. E. D.]

PROP. III. THEOR.

If two planes cut one another, their common section is a straight line.



Let two planes AB, BC cut one another, and let DB be their com. section: DB shall be a |.

Post. I. If it be not, from the p[t] D to B, draw, in the plane AB, the | DEB, and in the plane BC, the | DFB: these two |[s] DEB, DFB have the same extremities,

and ∴ they include a space betwixt them;

Ax.10.1. but this is impossible;

∴ BD, the com. section of the planes AB, BC cannot but be a |.

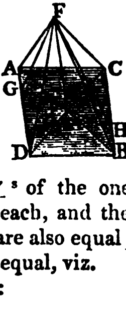
∴ *if two planes, &c.* [Q. E. D.]

PROP. IV. THEOR.

If a straight line stand at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

Let the | EF stand at r^t $\angle^s$ to each of the |^s AB, CD, in E, the p^t of their intersection: EF shall also be at r^t $\angle^s$ to the plane passing through AB, CD.

Take the |^s AE, EB, CE, ED, all = one another; through E draw, in the plane in w^h are AB, CD, any | GEH: join AD, CB; and from any p^t F, in EF, draw FA, FG, FD, FC, FH, FB:



then, in $\triangle^s$ AED, CEB,

∵ { side AE = BE, ED = EC,
and also $\angle$ AED = $\angle$ CEB, 15. 1.

∴ { base AD = base CB,
and $\angle$ DAE = $\angle$ EBC: 4. 1.

But, also $\angle$ AEG = $\angle$ BEH: 15 1.

∴ the $\triangle^s$ AEG, BEH have two $\angle^s$ of the one = two $\angle^s$ of the other, each to each, and the sides AE, EB, adj^t to the equal $\angle^s$, are also equal;

∴ the other sides of the $\triangle^s$ are equal, viz. 26. 1.

GE = EH, AG = BH:

And ∵ AE = EB,

and FE is com. and at r^t $\angle^s$ to AB,

∴ base AF = FB; 4. 1.

for the same reason, CF = FD:

hence, in the $\triangle^s$ FAD, FCB,

$\because \begin{cases} \text{side } AF = FB,\ AD = CB, \\ \text{and also base } FD = FC, \end{cases}$

8. 1. $\therefore \angle FAD = \angle FBC$:

and in $\triangle^s$ FAG, FBH,

$\because \begin{cases} \text{side } FA = FB,\ AG = BH, \\ \text{and also } \angle FAG = \angle FBH: \end{cases}$

4. 1. $\therefore$ base FG = base FH:

hence, in $\triangle^s$ FEG, FEH,

$\because \begin{cases} \text{side } EG = EH,\ EF \text{ is com.}, \\ \text{and also base } FG = FH; \end{cases}$

8. 1. $\therefore \angle GEF = \angle HEF$,

Def. 10. 1. and $\therefore$ each of these $\angle^s$ is a r^t $\angle$:

$\therefore$ FE makes r^t $\angle^s$ with GH,

i.e. with any | drawn through E in the plane passing through AB, CD.

In like manner, it may be proved, that FE makes r^t $\angle^s$ with every | w^h meets it in that plane.

Def. 3. 11. But a | is at r^t $\angle^s$ to a plane, when it makes r^t $\angle^s$ with every | w^h meets it in that plane;

$\therefore$ EF is at r^t $\angle^s$ to the plane in w^h are AB, CD.

$\therefore$ *if a straight line, &c.* [Q. E. D.]

PROP. V. THEOR.

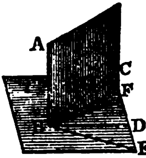
If three straight lines meet all in one point, and a straight line stands at right angles to each of them in that point; these three straight lines are in one and the same plane.

Let the | AB stand at r^t $\angle^s$ to each of the

|^s BC, BD, BE, in B, the p^t in w^h they meet: BC, BD, BE shall be in one and the same plane.

If not, let, if it be possible, BD, BE be in one plane, and BC be above it; and let a plane pass through AB, BC, the com. section of w^h with the plane in w^h

BD, DE are, is a |; 3. 11.



let this | be BF: then,

the three |^s AB, BC, BF are all in one plane,

viz. that w^h passes through AB, BC:

and $\because$ AB stands at r^t $\angle^s$ to each of the |^s BD, BE, 4. 11.

$\therefore$ it is at r^t $\angle^s$ to the plane passing through them; Def. 3. 11.

and $\therefore$ it makes r^t $\angle^s$ with every | w^h meets it in that plane:

but BF, w^h is in that plane, meets it;

and $\therefore$ $\angle$ ABF is a r^t $\angle$:

but, by hyp., $\angle$ ABC is also a r^t $\angle$;

$\therefore$ $\angle$ ABF = $\angle$ ABC,

and these $\angle^s$ are both in the same plane;

w^h is impossible:

$\therefore$ the | BC is not above the plane in w^h are BD, BE; Ax. 9.

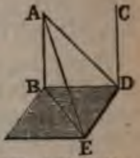
and $\therefore$ the three |^s BC, BD, BE are in one and the same plane.

$\therefore$ *if three straight lines, &c.* [Q. E. D.]

PROP. VI. THEOR.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Let the $|^s$ AB, CD be at $r^t \angle^s$ to the same plane: AB shall be $\parallel$ CD.



Let them meet the plane in the p^{ts} B, D, draw the $|$ BD, to w^h
11. 1. draw DE at $r^t \angle^s$, in the same
3. 1. plane; make DE = AB, and join BE, AE, AD:
then, $\because$ AB is $\perp$ to the plane,
Def. 3. 11. $\therefore$ it shall make $r^t \angle^s$ with every $|$ w^h meets it, and is in that plane:

but BD, BE, w^h are in that plane, do both meet AB; and $\therefore$ each of the $\angle^s$ ABD, ABE is a $r^t \angle$:

for the same reason, each of the $\angle^s$ CDB, CDE is a $r^t \angle$:

hence, in the $\triangle^s$ ABD, BDE,

$\because \begin{cases} \text{side AB} = \text{DE, BD is com.} \\ \text{and } r^t \angle \text{ ABD} = r^t \angle \text{ BDE,} \end{cases}$

4. 1. $\therefore$ base AD = base BE:

again, in $\triangle^s$ ABE, ADE,

$\because \begin{cases} \text{side AB} = \text{DE, BE} = \text{AD,} \\ \text{and base AE is com. to both,} \end{cases}$

8. 1. $\therefore$ $\angle$ ABE = $\angle$ ADE:
but ABE is a $r^t \angle$;
$\therefore$ ADE is also a $r^t \angle$,
and ED is $\perp$ to DA:

but it is also $\perp$ to each of the two BD, DC; and $\therefore$ ED is at $r^t \angle^s$ to each of the three $|^s$ BD, DA, DC, in the p^t in w^h they meet:
5. 11. $\therefore$ these three $|^s$ are all in the same plane:
2. 11. but AB is in the plane in w^h are BD, DA:
$\therefore$ AB, BD, DC are in one plane:

and each of the ∠ˢ ABD, BDC is a rᵗ ∠;
∴ AB is ∥ CD. 28. I.

∴ *if two straight lines, &c.* [Q. E. D.]

PROP. VII. THEOR.

If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels.

Let AB, CD be ∥ˢ, and take any pᵗ E in the one, and any pᵗ F in the other: the | wʰ joins E and F shall be in the same plane with the ∥ˢ.

If not, let it be, if possible, above the plane, as EGF; and in the plane ABCD in wʰ the ∥ˢ are, draw from E to F the | EHF: then,

∵ EGF is also a |,
∴ the two |ˢ EHF, EGF include a space between them; wʰ is impossible. Ax. 10. I

∴ the | joining the pᵗˢ E, F is not above the plane in wʰ the ∥ˢ AB, CD are;
and ∴ it is in that plane.

∴ *if two straight lines, &c.* [Q. E. D.]

PROP. VIII. THEOR.

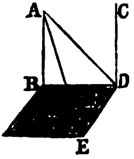
If two straight lines be parallel, and one of them be at right angles to a plane; the other also shall be at right angles to the same plane.

Let AB, CD be $\|^s$, and let one of them AB be at $r^t \angle^s$ to a plane: the other CD shall be at $r^t \angle^s$ to the same plane.

Let AB, CD meet the plane in the p^{ts} B, D, and
7. 11. join BD: then AB, CD, BD are in one plane.

11. 1. In the plane to w^h AB is at $r^t \angle^s$, draw DE at
3. 1. $r^t \angle^s$ to BD, make DE = AB, and join BE, AE, AD:

then, $\because$ AB is $\perp$ to the plane,
Def. 3. 11. $\therefore$ it is $\perp$ to every | w^h meets it, and is in that plane;
and $\therefore$ each of the $\angle^s$ ABD, ABE is a $r^t \angle$:
and $\because$ | BD meets the $\|^s$ AB, CD,
29. 1. $\therefore$ $\angle^s$ (ABD + CDB) = two $r^t \angle^s$:
but ABD is a $r^t \angle$;
$\therefore$ also CDB is a $r^t \angle$,
and CD is $\perp$ to BD:



and, in the $\triangle^s$ ABD, BDE,
$\because \left\{ \begin{array}{l} \text{side AB} = \text{DE, BD is com.,} \\ \text{and } r^t \angle \text{ ABD} = r^t \angle \text{ BDE,} \end{array} \right.$
4. 1. $\therefore$ base AD = base BE:

again, in the $\triangle^s$ ABE, ADE,
$\because \left\{ \begin{array}{l} \text{side AB} = \text{DE, BE} = \text{AD,} \\ \text{and the base AE is com.;} \end{array} \right.$
8. 1. $\therefore$ $\angle$ ABE = $\angle$ ADE:

but ABE is a $r^t \angle$;
and $\therefore$ ADE is also a $r^t \angle$,
and ED is $\perp$ to AD : Constr.
but it is also $\perp$ to BD;
ED is $\perp$ to the plane w^h passes through BD, AD; 4. 11.
$\therefore$ makes $r^t \angle^s$ with every | meeting it in that plane: Def. 3 11.

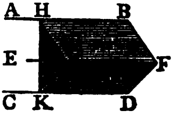
t DC is in the plane passing through BD, DA,
ll three are in the plane in w^h are the $||^s$ AB, CD;
$\therefore$ ED is at $r^t \angle^s$ to DC,
and $\therefore$ CD is at $r^t \angle^s$ to DE:
but CD is also at $r^t \angle^s$ to DB;
$\therefore$ CD is at $r^t \angle^s$ to the two $|^s$ DE, DB,
in the p^t D of their intersection;
$\therefore$ is at $r^t \angle^s$ to the plane passing through 4. 11.
, DB, w^h is the same plane to w^h AB is at $r^t \angle^s$.

. *if two straight lines, &c.* [Q. E. D.]

PROP. IX. THEOR.

) straight lines which are each of them parallel to e same straight line, and not in the same plane ith it, are parallel to one another.

et AB, CD be each of them || EF, and not e same plane with it: AB shall be || CD.



1 EF take any p^t G, from
lraw, in the plane passing
ugh EF, AB, the | GH at
^{s} to EF; and in the plane 11. 1.
ing through EF, CD, draw

GK at $r^t \angle^s$ to the same EF : then,
$\because$ EF is $\perp$ both to GH and GK,
4. 11. $\therefore$ EF is $\perp$ to the plane HGK passing through them:
and EF is $\parallel$ AB :
8. 11. $\therefore$ AB is at $r^t \angle^s$ to the plane HGK.

For the same reason,
CD is also at $r^t \angle^s$ to the plane HGK.
$\therefore$ AB, CD are each of them at $r^t \angle^s$ to the plane HGK.

But if two $|^s$ are at $r^t \angle^s$ to the same plane, they
6. 11. are $\parallel$ to one another:
$\therefore$ AB is $\parallel$ CD.

$\therefore$ *if two straight lines, &c.* [Q. E. D.]

PROP. X. THEOR.

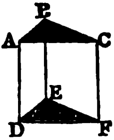
If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two, the first two and the other two shall contain equal angles.

Let the two $|^s$ AB, BC, w^h meet one another, be $\parallel$ the two $|^s$ DE, EF, w^h meet one another, and are not in the same plane with AB, BC :

then shall $\angle$ ABC = $\angle$ DEF.

Take BA, BC, ED, EF all = one another; and join AD, CF, BE, AC, DF : then,

∵ BA is = and ∥ ED,
∴ AD is = and ∥ BE. 33. 1



For the same reason,
CF is = and ∥ BE,
∴ AD, CF are each = and ∥ BE.

But |s that are ∥ the same |, and not in the same plane with it, are ∥ one another: 9. 11.
∴ AD is ∥ CF;
and AD = CF: Ax. 1.1.
and AC, DF join them towards the same parts;
∴ AC is = and ∥ DF. 33. 1.

Hence, in △s ABC, DEF,
∵ { side AB = DE, BC = EF,
and also base AC = base DF,
∴ ∠ ABC = ∠ DEF. 8 1.

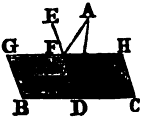
∴ *if two straight lines, &c.* [Q. E. D.]

PROP. XI. PROB.

To draw a straight line perpendicular to a plane from a given point above it.

Let A be the given p[t] above the plane BH: it is req[d] to draw from A a ⊥ to the plane BH.

In the plane draw any | BC, and from A draw AD ⊥ to BC: if then AD be also ⊥ to the plane BH, 12. 1



the thing req[d] is already done: but if this be not the case, from D draw, in the plane BH,
11. 1. the | DE, at r^t $\angle^s$ to BC; and from A draw AF $\perp$ to DE: AF shall be $\perp$ to the plane BH.

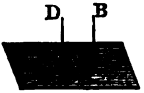
31. 1. Through F draw GH || BC: then,
∵ BC is at r^t $\angle^s$ to ED, DA,
4. 11. ∴ BC is at r^t $\angle^s$ to the plane passing through ED, DA:
and GH is || BC;
but if two $|^s$ be ||, and one be at r^t $\angle^s$ to a plane,
8. 11. the other is at r^t $\angle^s$ to the same plane;
∴ GH is at r^t $\angle^s$ to the plane through ED, DA,
Def. 3. 11. and ∴ GH is $\perp$ to every | meeting it in that plane:
but AF, w^h is in that plane meets it;
∴ GH is $\perp$ to AF:
and ∴ AF is $\perp$ to GH:
and AF is $\perp$ to DE;
∴ AF is $\perp$ to each of the $|^s$ GH, DE.

But if a | stand at r^t $\angle^s$ to each of two $|^s$ in the p^t of their intersection,
4. 11. it is at r^t $\angle^s$ to the plane passing through them:
but the plane through ED, GH is the plane BH;
∴ AF is $\perp$ to the plane BH.

∴ *from the given point* A, *above the plane* BH, *the straight line* AF *is drawn perpendicular to that plane.* [Q. E. F.]

PROP. XII. PROB.

To erect a straight line at right angles to a given plane, from a point given in the plane.



Let A be the p^t given in the plane : it is req^d to erect a | from the p^t A at r^t $\angle^s$ to the plane.

From any p^t B above the plane draw BC $\perp$ to it ; and from A 11. 11.
draw AD || BC : then, 31. 1.

∵ the |^s AD, CB are ||^s,

and one of them BC is at r^t $\angle^s$ to the given plane,

∴ the other AD is also at r^t $\angle^s$ to it : 8. 11.

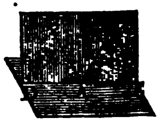
∴ *a straight line has been erected at right angles to a given plane, from a point given within it.*

[Q. E. F.]

PROP. XIII. THEOR.

From the same point in a given plane there cannot be two straight lines at right angles to the plane upon the same side of it : and there can be but one perpendicular to a plane from a point above the plane.

For, if it be possible, let the two |^s AB, AC be at r^t $\angle^s$ to a given plane from the same p^t A in the plane, and upon the same side of it.



Let a plane pass through AB, AC; the com. section of
3. 11. this with the given plane is a |
passing through A: let DAE
be this com. section: then,

the $|^s$ AB, AC, ED are in one plane:
and ∵ CA is at r^t $\angle^s$ to the given plane,
Def. 3. 11. ∴ it makes r^t $\angle^s$ with every |
meeting it in that plane:
but DAE, w^h is in that plane, meets CA;
∴ CAE is a r^t $\angle$:

for the same reason,

BAE is a r^t $\angle$:
Ax. 11. ∴ $\angle$ CAE $=$ $\angle$ BAE;
and they are in one plane;
w^h is impossible.

Also, from a p^t above a plane, there can be but
one ⊥ to that plane: for,
if there could be two,
they would be ∥ one another;
6. 11. w^h is absurd.

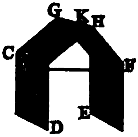
∴ *from the same point, &c.* [Q. E. D.]

PROP. XIV. THEOR.

Planes to which the same straight line is perpendicular, are parallel to one another.

Let the | AB be ⊥ to each of the planes CD, EF: these planes shall be ∥ one another.

If not, they shall, when prod[d], meet one another: let them meet; and let their com. section be the | GH, in w[h] take any p[t] K, and join AK, BK: then,



∵ AB is ⊥ to the plane EF,

∴ it is ⊥ to the | BK, w[h] is in that plane; Def. 3. 11.

and ∴ ABK is a r[t] ∠:

For the same reason,

BAK is a r[t] ∠:

∴ the two ∠[s] (ABK + BAK) = two r[t] ∠[s],

i.e. two ∠[s] of a △ = two r[t] ∠[s],

w[h] is impossible. 17. 1.

∴ the planes CD, EF, though prod[d], do not meet, Def. 8. 11.

i.e. they are || to one another.

∴ *planes, &c.* [Q. E. D.]

PROP. XV. THEOR.

If two straight lines meeting one another be parallel to two other straight lines which meet one another, but are not in the same plane with the first two, the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC, two |[s] meeting one another, be || to two |[s] DE, EF, that meet one another, but are not in the same plane with AB, BC: the planes through AB, BC, and DE, EF shall not meet, though prod[d]

11. 11. From B draw BG $\perp$ to the plane w^h passes through DE, EF, and let it meet that plane in G;
31. 1. and through G draw GH $\parallel$ ED, and GK $\parallel$ EF: then,

$\because$ BG is $\perp$ to the plane through DE, EF,

Def. 3. 11. $\therefore$ it makes $r^t \angle^s$ with every | meeting it in that plane:

but the $|^s$ GH, GK in that plane meet it;

$\therefore$ each of the $\angle^s$ BGH, BGK is a $r^t \angle$:

9. 11. and $\because$ BA is $\parallel$ GH,

(for each of them is $\parallel$ DE,

and they are not both in the same plane with it),

29. 1. $\therefore \angle^s$ (GBA + BGH) = two $r^t \angle^s$:

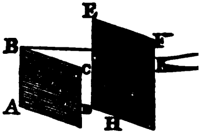
and BGH is a $r^t \angle$;

$\therefore$ also GBA is a $r^t \angle$,

and GB $\perp$ to BA:

for the same reason,

GB is $\perp$ to BC.



Hence,

$\because$ the | GB stands at $r^t \angle^s$ to the two $|^s$ BA, AC, that cut one another in B,

4. 11. $\therefore$ GB is $\perp$ to the plane through BA, BC:

Constr. and it is $\perp$ to the plane through DE, EF;

$\therefore$ GB is $\perp$ to each of the planes through AB, BC, and DE, EF:

but planes, to w^h the same | is $\perp$,

14. 11. are $\parallel$ one another;

$\therefore$ the plane through AB, BC is $\parallel$ that through DE, EF.

$\therefore$ *if two straight lines, &c.* [Q. E. D.]

PROP. XVI. THEOR.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the || planes AB, CD be cut by the plane EFHG, and let their com. sections with it be EF, HG: EF shall be || GH.

For, if it is not, EF, GH shall meet, if prod[d], either on the side of FH, or EG.

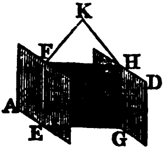
First, let them be prod[d] on the side of FH, and if possible, meet in the p[t] K: then,

∵ EFK is in the plane AB,

∴ every p[t] in EFK is in that plane: 1. 11

and K is a p[t] in EFK;

∴ K is in the plane AB:



for the same reason,

K is also in the plane CD:

∴ the planes AB, CD, prod[d], meet one another:

but, by hyp[s], these planes are ||,

and ∴ do not meet one another:

∴ the |[s] EF, GH, do not meet when prod[d] on the side of FH.

In the same manner it may be proved, that EF, GH do not meet, when prod[d], on the side of EG.

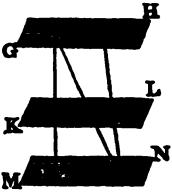
But |[s] w[h] are in the same plane, and do not meet though prod[d] either way, are ||;

and ∴ EF is || GH.

∴ *if two parallel planes, &c.* [Q. E. D.]

PROP. XVII. THEOR.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.



Let the $|^{s}$ AB, CD be cut by the || planes GH, KL, MN, in the p^{ts} A, E, B; C, F, D: then, AE : EB :: CF : FD.

Join AC, BD, AD; let AD meet the plane KL in X; and join EX, XF: then,

∵ the || planes KL, MN are cut by the plane EBDX,

16. 11. ∴ the com. sections EX, BD are ||:

for the same reason,

∵ the || planes GH, KL are cut by the plane AXFC,

∴ the com. sections AC, XF are ||:

and ∵ EX is || BD, a side of △ ABD,

2. 6. ∴ AE : EB :: AX : XD:

again, ∵ XF is || AC, a side of △ ADC,

∴ AX : XD :: CF : FD:

and it was proved, that

AX : XD :: AE : EB;

11. 5. ∴ AE : EB :: CF : FD.

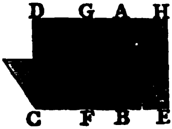
∴ *if two straight lines, &c.* [Q. E. D.]

PROP. XVIII. THEOR.

If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let the | AB be at $r^t \angle^s$ to the plane CK: every plane w^h passes through AB shall be at $r^t \angle^s$ to the plane CK.

Let any plane DE pass through AB, and let CE be the com. section of the planes DE, CK; take any p^t F in CE, from w^h draw FG in the plane DE at $r^t \angle^s$ to CE: then, 11. 1.



∵ AB is ⊥ to the plane CK,
∴ it is also ⊥ to every | in that plane meeting it; Def. 3. 11.
and ∴ it is ⊥ to CE:
∴ ABF is a $r^t \angle$;
but GFB is also a $r^t \angle$: Constr
∴ AB is ∥ FG: 28. 1.
and AB is at $r^t \angle^s$ to the plane CK;
∴ FG is also at $r^t \angle^s$ to the same plane. 8. 11.

But one plane is at $r^t \angle^s$ to another plane when the $|^s$ drawn in one of the planes, at $r^t \angle^s$ to their com. section, are also at $r^t \angle^s$ to the other plane: Def. 4. 11.
and it has been proved that any | FG in the plane DE, w^h is at $r^t \angle^s$ to CE, the com. section of the planes, is ⊥ to the other plane CK;

∴ the plane DE is at $r^t \angle^s$ to the plane CK.

In like manner, it may be proved that all planes w^h pass through AB are at $r^t \angle^s$ to the plane CK.

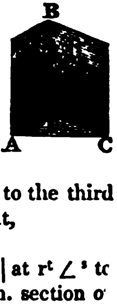
∴ *if a straight line, &c.* [Q. E. D.]

PROP. XIX. THEOR.

If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let the two planes AB, BC be each of them $\perp$ to a third plane, and let BD be the com. section of the first two: BD shall be $\perp$ to the third plane.

11. 1. If it be not, from the p^t D draw, in the plane AB, the $|$ DE at r^t $\angle^s$ to AD the com. section of the plane AB with the third plane; and in the plane BC draw DF at r^t $\angle^s$ to CD the com. section of the plane BC with the third plane: then, $\because$ the plane AB is $\perp$ to the third plane, and DE is drawn in the plane AB at r^t $\angle^s$ to AD, their com. section;



Def. 4. 11. $\therefore$ DE is $\perp$ to the third plane.

In like manner it may be proved, that DF is $\perp$ to the third plane;

i.e. from the p^t D two $|^s$ stand at r^t $\angle^s$ to the third plane, upon the same side of it,

13. 11. w^h is impossible:

$\therefore$ from the p^t D there cannot be any $|$ at r^t $\angle^s$ to the third plane, except BD the com. section of the planes AB, BC:

$\therefore$ BD is $\perp$ to the third plane.

$\therefore$ *if two planes, &c.* [Q. E. D.]

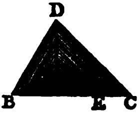
PROP. XX. THEOR.

If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let the solid $\angle$ at A be contained by the three plane $\angle^s$ BAC, CAD, DAB: any two of them shall be $>$ the third.

If the $\angle^s$ BAC, CAD, DAB be all equal, it is evident that any two of them are $>$ the third.

But, if they are not, let BAC be that $\angle$ w^h is $\nless$ either of the other two, and is $>$ one of them DAB: at the p^t A in the | AB, and in the plane w^h passes through AB, AC, make $\angle$ BAE $= \angle$ DAB; and take AE $=$ AD; through E draw BEC cutting AB, AC in the p^{ts} B, C, and join DB, DC: 23. 1.



then in the $\triangle^s$ BAD, BAE,

$\because \begin{cases} \text{side AD} = \text{AE, AB is com.,} \\ \text{and } \angle \text{ BAD} = \angle \text{ BAE;} \end{cases}$

$\therefore$ the base DB $=$ the base BE; 4. 1.

and $\because$ BD, DC are together $>$ CB, 20. 1.

and it has been proved that one of them BD $=$ BE, a part of CB,

$\therefore$ the other DC is $>$ the rem^g part EC: Ax. 5.

Hence, in the $\triangle^s$ DAC, EAC,

$\because \begin{cases} \text{side DA} = \text{EA, AC is com.} \\ \text{but base DC} > \text{base EC;} \end{cases}$

$\therefore \angle$ DAC $> \angle$ EAC; 25. 1.

and $\angle$ DAB $=$ $\angle$ BAE;

Ax. 4. $\therefore$ $\angle^s$(DAB + DAC) $>$ $\angle^s$(BAE + EAC),

i.e. $>$ $\angle$ BAC

but $\angle$ BAC is $\not<$ either of the $\angle^s$ DAB, DAC:

$\therefore$ $\angle$ BAC + either of these $\angle^s$ is $>$ the other.

$\therefore$ *if a solid angle &c.* [Q. E. D.]

PROP. XXI. THEOR.

Every solid angle is contained by plane angles, which together are less than four right angles.

First, let the solid angle at A be contained by the three plane $\angle^s$ BAC, CAD, BAD: these three together shall be $<$ four r^t $\angle^s$.

Take in each of the $|^s$ AB, AC, AD, any p^{ts} B, C, D; and join BC CD, DB: then,

$\because$ the solid angle at B is contained by the three plane $\angle^s$ CBA, ABD, DBC,

20. 11. $\therefore$ any two of them are $>$ the third;

and $\therefore$ $\angle^s$(CBA+ABD) are $>$ $\angle$ DBC:

for the same reason,

$\angle^s$(BCA+ACD) are $>$ $\angle$ DCB:

and $\angle^s$(CDA+ADB) are $>$ $\angle$ BDC:

$$\therefore \left\{\begin{array}{c}\text{the six } \angle^s \\ (\text{CBA} + \text{ABD} \\ + \text{BCA} + \text{ACD} \\ + \text{CDA} + \text{ADB})\end{array}\right\} \text{are} > \left\{\begin{array}{c}\text{the three } \angle^s \\ (\text{DBC} + \text{DCB} \\ + \text{BDC}):\end{array}\right.$$

but the three $\angle^s(DBC+DCB+BDC)=$ two $r^t \angle^s$; 32. 1.

$$\therefore \text{ the six } \angle \left\{\begin{array}{c} CBA+ABD \\ +BCA+ACD \\ +CDA+ADB \end{array}\right\} \text{ are } > \text{ two } r^t \angle^s:$$

but, $\because$ the three $\angle^s$ of every $\triangle =$ two $r^t \angle^s$:

$\therefore$ the nine $\angle^s$ of the three $\triangle^s$ ABC, ACD, ADB

$$\text{viz. the } \angle^s \left\{\begin{array}{c} CBA+ABD+BCA \\ +ACD+CDA+ADB \\ +BAC+BAD+CAD \end{array}\right\} = \text{six } r^t \angle^s;$$

and of these nine $\angle^s$, it has been shown that the first six are $>$ two $r^t \angle^s$,

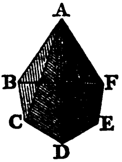
$\therefore$ the last three $\angle^s$, viz.

the $\angle^s$ $(BAC+BAD+CAD)$ are $<$ four $r^t \angle^s$;

and these three $\angle^s$ contain the solid angle at A.

Next, let the solid angle at A be contained by any n^o of plane $\angle^s$ BAC, CAD, DAE, EAF, FAB: these shall together be $<$ four $r^t \angle^s$.

Let the planes in w^h the $\angle^s$ are be cut by a plane, and let the com. sections of it with those planes be BC, CD, DE, EF, FB: then,



$\because$ the solid angle at B is contained by three plane $\angle^s$ CBA, ABF, FBC, 20. 11

of w^h any two are $>$ the third;

$\therefore$ the $\angle^s$ (CBA, ABF) are $>$ $\angle$ FBC:

for the same reason,

the two plane $\angle^s$ at each of the p^{ts} C, D, E, F, viz. those $\angle^s$ w^h are at the bases of the $\triangle^s$ having the com. vertex A,

are together $>$ the third $\angle$ at the same p^t, w^h is one of the $\angle^s$ of the polygon BCDEF:

$\therefore$ all the $\angle^s$ at the bases of the $\triangle^s$ are together $>$ all the $\angle^s$ of the polygon:

and,

32. 1. $\because$ all the $\angle^s$ of the $\triangle^s$ $=$ twice as many $r^t \angle^s$ as there are $\triangle^s$, *i. e.* as there are sides of the polygon,

and that likewise

Cor 1. 32. .. all the $\angle^s$ of the polygon $+$ four $r^t \angle^s$ $=$ twice as many $r^t \angle^s$ as there are sides in the polygon;

Ax. 1. $\therefore$ all the $\angle^s$ of the $\triangle^s$ $=$ all the $\angle^s$ of the polygon $+$ four $r^t \angle^s$:

but it has been proved that

all the $\angle^s$ at the bases of the $\triangle^s$ are $>$ all the $\angle^s$ of the polygon;

$\therefore$ the rem^g $\angle^s$ of the $\triangle^s$, viz. those at the vertex are $<$ four $r^t \angle^s$;

and these rem^g $\angle^s$ contain the solid angle at A.

$\therefore$ *every solid angle, &c.* [Q. E. D.]

*** The remainder of this Book is seldom read in the University.

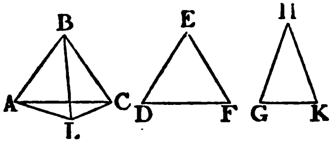
PROP. XXII. THEOR.

If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal; a triangle may be made of the straight lines that join the extremities of those equal straight lines.

Let ABC, DEF, GHK be the three plane $\angle^s$, whereof every two are $>$ the third, and let them

be contained by the equal |s AB, BC, DE, EF, GH, HK: if their extremities be joined by the |s AC, DF, GK, a △ may be made of three |s wh are = AC, DF, GK; *i. e.* every two of them shall together be > the third.

If the ∠s at B, E, H be equal,
the |s AC, DF, GK are also equal; 4. 1
and ∴ any two of them > the third:
but if the ∠s be not all equal,
let the ∠ ABC be ≮ either of the two at E, H;
then the | AC is ≮ either of the other two DF, GK; 4 or 24. 1.
and ∴ it is plain that
AC + either of the other two must be > the third.



Also, DF + GK shall be > AC.
For, at the pt B in the | AB, 23. 1.
make ∠ ABL = ∠ GHK, take BL = one of the |s AB, BC, DE, EF, GH, HK, and join AL, LC:
then, in △s ABL, GHK,
∵ { side AB = GH, BL = HK,
and ∠ ABL = ∠ GHK;
∴ the base AL = the base GK: 4. 1.

and,
∵ the ∠s at E, H are together > the ∠ ABC, Hyp.
of wh, the ∠ at H = ∠ ABL,
∴ the remg ∠ at E is > the ∠ LBC: Ax. 5.

hence, in the △s BLC, EDF,

∵ { side LB = DE, BC = EF,
but ∠ DEF is > ∠ LBC,

24. 1. ∴ the base DF is > the base LC:

and it has been proved that

GK = AL;

Ax. 4. ∴ DF and GK are > LC and AL:

20. 1. but LC and AL are > AC;

à fortiori ∴ DF and GK are > AC.

∴ every two of these |s AC, DF, GK,
are > the third.

And ∴ *a triangle may be made, the sides of which*
22. 1 *shall be equal to* AC, DF, GK. [Q. E. D.]

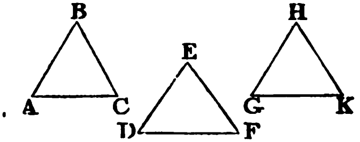
PROP. XXIII. PROB.

To make a solid angle which shall be contained by three given plane angles, any two of them being
20. 11. *greater than the third, and all three together*
21. 11. *less than four right angles.*

Let ABC, DEF, GHK be the three given plane ∠s, of wh any two are > the third, and all of them together < four rt ∠s. It is reqd to make a solid angle contained by three plane ∠s, wh are = ABC, DEF, GHK, each to each.

From the |s wh contain the ∠s cut off AB, BC, DE, EF, GH, HK, all = one another; and join

AC, DF, GK. Then a △ may be made of three |s 22. 1



wh are = AC, DF, GK: let LMN be this △, so 22. 1.
that AC = LM, DF = MN, GK = LN; about
the △ LMN desc. a ⊙, and find its cent. X, wh 5. 4.
will be either within the △, or in one of its sides, 1. 3.
or without it.

First, let the cent. X be within the △, and join LX, MX, NX: AB shall be > LX.

If not, AB must be either = or < LX:
first, let these |s be equal: then,

∵ AB = LX,

and that also

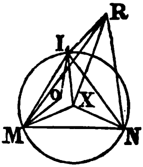
AB = BC and LX = XM,

∴ AB, BC = LX, XM,

each to each;

and the base AC = the base LM; Constr.

∴ ∠ ABC = ∠ LXM. 8. 1.



For the same reason,
∠ DEF = ∠ MXN, and ∠ GHK = ∠ NXL:
and ∴

$$\left\{ \angle^s (ABC + DEF + GHK) \right\} = \left\{ \angle^s (LXM + MXN + NXL). \right\}$$

but
the three ∠s (LXM + MXN + LXN) = four rt ∠s; Cor. 2.
∴ also the three (ABC + DEF + GHK) = four rt ∠s: 15. 1.

but, by the hyp^s, these $\angle^s$ are $<$ four r^t $\angle^s$;

wh is absurd:

$\therefore$ AB is $\neq$ LX.

But neither can AB be $<$ LX: for, if possible, let it be less; and on the | LM, on that side of it

22. 1. on wh is the cent. X, desc. the $\triangle$ LOM, of wh the sides LO, OM $=$ AB, BC, each to each: then,

$\because$ the base LM $=$ the base AC,

8. 1. $\therefore$ $\angle$ LOM $= \angle$ ABC:

And, by hyp^s, AB, *i.e.* LO, is $<$ LX:

$\therefore$ LO, OM fall within the $\triangle$ LXM;

21. 1. for, if they fell upon its sides, or without it, they would be $=$ or $>$ LX, XM:

21. 1. $\therefore \angle$ LOM, *i. e.* $\angle$ ABC, is $> \angle$ LXM.

In the same manner it may be proved that $\angle$ DEF is $> \angle$ MXN, and $\angle$ GHK $> \angle$ NXL

$$\therefore \left\{ \begin{array}{c} \angle^s(\text{ABC}+\text{DEF} \\ +\text{GHK}) \end{array} \right\} > \left\{ \begin{array}{c} \angle^s(\text{LXM}+\text{MXN} \\ +\text{NXL}) \end{array} \right\}$$

Cor. 2. 15. 1. *i.e.* $>$ four r^t $\angle^s$:

Hyp. but the same three $\angle^s$ are also $<$ four r^t $\angle^s$:

wh is absurd:

$\therefore$ AB is $\not<$ LX:

and it has been proved that

AB is $\neq$ LX;

$\therefore$ AB is $>$ LX.

Next, let the cent. X of the $\odot$ fall in one of the sides of the $\triangle$, viz. in MN, and join XL.

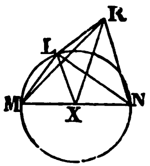
In this case, also, AB shall be $>$ LX.

If not, AB is either $=$ or $<$ LX.

First, let AB $=$ LX;

then $AB + BC = LX + MX$

or $DE + EF = MN$:



but, by the constr^n,

$MN = DF$;

$\therefore DE + EF = DF$,

w^h is impossible: 20. 1.

$\therefore$ AB is $\neq LX$:

nor is $AB < LX$;

for then, much more, an absurdity would follow:

$\therefore$ AB is $> LX$.

But, let the cent. X of the $\odot$ fall without the $\triangle LMN$, and join LX, MX, NX.

In this case likewise, AB shall be $> LX$.

If not, it is either $=$ or $< LX$.

First, let $AB = LX$: then it may be proved, as in the first case, that

$\angle ABC = \angle MXL$, and $\angle GHK = \angle LXN$:

$\therefore$ the whole $\angle MXN =$ the two $\angle^s(ABC + GHK)$:

but $\angle^s(ABC + GHK)$ are $> \angle DEF$; Hyp.

$\therefore$ also $\angle MXN$ is $> \angle DEF$:

but, $\because$ DE, EF $=$ MX, XN, each to each,

and the base $DF =$ the base MN, 8. 1.

$\therefore \angle MXN = \angle DEF$:

but, from above,

$\angle MXN$ is $> \angle DEF$;

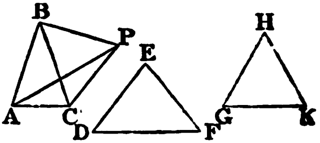
w^h is absurd.

$\therefore AB$ is $\neq LX$.

Nor yet is $AB < LX$: for then, as has been proved in first case,

$\angle ABC$ is $> \angle MXL$, and $\angle GHK > \angle LXN$.

At the p[t] B, in the | CB, make ∠ CBP = ∠ GHK, take BP = HK, and join CP, AP: then,



∵ CB = GH,
∴ CB, BP = GH, HK, each to each;
and they contain equal ∠ˢ;
∴ the base CP = the base GK, *i.e.* LN.

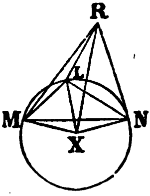
And in the isosc. △ˢ ABC, MXL,
∵ ∠ ABC is > ∠ MXL,
32. 1. ∴ ∠ MLX is > ∠ ACB.

In like manner,
∵ ∠ GHK or CBP is > ∠ LXN,
∴ ∠ XLN is > ∠ BCP:
and ∴ the whole ∠ MLN is > the whole ∠ ACP

And,
∵ the sides ML, LN = AC, CP, each to each,
but, ∠ MLN is > ∠ ACP,
24. 1. ∴ the base MN is > the base AP:
but MN = DF;
∴ also, DF is > AP.

Again,
∵ DE, EF = AB, BP, each to each,
but base DF is > base AP,
25. 1. ∴ ∠ DEF is > ∠ ABP:

but $\angle$ ABP $= \angle^s$ (ABC + CBP),
i.e. $= \angle^s$ (ABC + GHK):
$\therefore \angle$ DEF is $> \angle^s$ (ABC + GHK):
but it is also $<$ these $\angle^s$, Hyp.
w^h is impossible;
$\therefore$ AB is $\not<$ LX:
and it has been proved that
AB is $\neq$ LX:
$\therefore$ AB is $>$ LX.



From the p^t X erect XR at $r^t \angle^s$ to the plane of the $\odot$ LMN. 12. 11.
And since it has been proved in all the cases, that AB is $>$ LX, find a sq. $=$ the excess of the sq. of AB above the sq. of LX, and make RX $=$ the side of this sq., and join RL, RM, RN. The solid angle at R shall be the angle req^d. For,

$\because$ RX is $\perp$ to the plane of the $\odot$ LMN,
$\therefore$ it is $\perp$ to each of the $|^s$ LX, MX, NX. Def. 3. 11.
And $\because$ side LX $=$ MX,
and XR is com. and at $r^t \angle^s$ to each,
$\therefore$ the base RL $=$ the base RM. 4. 1.
For the same reason,
RN $=$ each of the two RL, RM;
and $\therefore$ the three $|^s$ RL, RM, RN, are all equal.
And $\because$ $RX^2 = AB^2 - LX^2$, Constr.
$\therefore$ $AB^2 = RX^2 + LX^2$,
but, $\because$ LXR is a $r^t \angle$,
$\therefore$ $RL^2 = RX^2 + LX^2$; 47. 1.
$\therefore$ $AB^2 = RL^2$,
and AB $=$ RL.

But
AB $=$ each of the $|^s$ BC, DE, EF, GH, HK,
and RL $=$ each of the two RM, RN;

∴ each of the |s AB, BC, DE, EF, GH, HK, = each of the |s RL, RM, RN.

And,

∵ RL, RM = AB, BC, each to each,

Constr. and the base LM = the base AC,

8. 1. ∴ ∠ LRM = ∠ ABC.

For the same reason,

∠ MRN = ∠ DEF. ∠ NRL = ∠ GHK.

∴ *there is made a solid angle at* R, *which is contained by three plane angles* LRM, MRN, NRL, *which are equal to the three given plane angles* ABC, DEF, GHK, *each to each.* [Q. E. F.]

PROP. A. THEOR.

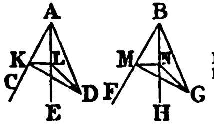
If each of two solid angles be contained by three plane angles, which are equal to one another, each to each; the planes in which the equal angles are have the same inclination to one another.

Let there be two solid angles at the pts A, B; and let the angle at A be contained by the three plane ∠s CAD, CAE, EAD; and the angle at B by the three plane ∠s FBG, FBH, HBG; of wh ∠ CAD = FBG, CAE = FBH, and EAD = HBG: the planes in wh the equal ∠s are shall have the same inclination to one another.

11. 1. In the | AC take any pt K, from K draw in the plane CAD the | KD at rt ∠s to AC, and in the plane CAE the | KL at rt ∠s to the same AC

then the $\angle$ DKL is the inclination of the plane CAD to the plane CAE. Def. 6 11.

In BF take BM = AK, and from the p^t M draw in the planes FBG, FBH, the $|^s$ MG, MN at r^t $\angle^s$ to BF; then the $\angle$ GMN is the inclination of the plane FBG to the plane FBH. Def. 6. 11.



Join LD, NG. Then, in the $\triangle^s$ KAD, MBG,

$\because$ $\left\{ \begin{array}{l} \angle \text{ KAD} = \text{MBG, r}^t \angle \text{ AKD} = \text{r}^t \angle \text{ BMG,} \\ \text{and also the adj}^t \text{ sides AK, BM are equal,} \end{array} \right.$ Hyp.

$\therefore$ side KD = MG, and AD = BG: 26. 1.

for the same reason, in the $\triangle^s$ KAL, MBN,

side KL = MN, and AL = BN:

hence, in the $\triangle^s$ LAD, NBG,

the sides LA, AD = NB, BG, each to each,

and they also contain equal $\angle^s$; 4. 1.

$\therefore$ the base LD = the base NG,

Lastly, in the $\triangle^s$ KLD, MNG,

the sides DK, KL = GM, MN, each to each,

and also the base LD = the base NG: 8. 1.

$\therefore$ $\angle$ DKL = $\angle$ GMN:

but $\angle$ DKL is the inclination of the plane CAD to the plane CAE, and $\angle$ GMN is the inclination of the plane FBG to the plane FBH,

$\therefore$ these planes have the same inclination to each other. Def. 7 11.

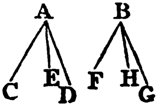
And in the same manner it may be dem^d that the other planes in w^h the equal $\angle^s$ are, have the same inclination to one another.

$\therefore$ *if two solid angles, &c.* [Q. E. D.]

PROP. B. THEOR.

If two solid angles be contained, each by three plane angles which are equal to one another, each to each, and alike situated; these solid angles are equal to one another.

Let there be two solid angles at A and B, of wʰ the solid angle at A is contained by the three plane ∠ˢ CAD, CAE, EAD; and that at B, by the three plane ∠ˢ FBG, FBH, HBG; of wʰ ∠ˢ



CAD = FBG,
CAE = FBH,
and EAD = HBG:

the solid angle at A shall be = the solid angle at B.

Let the solid angle at A be applied to that at B: and first, let the plane ∠ CAD be applied to the plane ∠ FBG, so that the pᵗ A may coincide with the pᵗ B, and the | AC with BF: then,

∵ ∠ CAD = ∠ FBG,
∴ AD must coincide with BG:

and ∵ the inclination of the plane CAE to the plane CAD is = the inclination of the plane FBH to the plane FBG,

and that the planes CAD, FBG are coincident;

A. 11. ∴ the plane CAE coincides with the plane FBH:

and ∵ the | AC coincides with BF,
and that ∠ CAE = ∠ FBH;
∴ AE coincides with BH:

and AD coincides with BG;
∴ the plane EAD coincides with the plane HBG:
∴ the solid angle at A coincides with that at B.

And ∴ *the angles are equal to one another.* Ax. 8.]
[Q. E. D.]

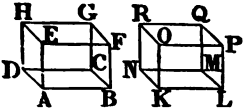
PROP. C. THEOR.

Solid figures which are contained by the same number of equal and similar planes alike situated, and having none of their solid angles contained by more than three plane angles, are equal and similar to one another.

Let AG, KQ be two solid figs contained by the same no of simr and equal planes, alike situated, viz. let the plane AC be simr and = the plane KM; the plane AF to KP, BG to LQ, GD to QN, DE to NO; and, lastly, FH to PR: the solid fig. AG shall be simr and = the solid fig. KQ.

For,
∵ the solid angle at A is contained by the three plane $\angle$s BAD, BAE, EAD,
and that these $\angle$s respectively = the plane $\angle$s LKN, LKO, OKN, wh contain the solid angle at K; Hyp.
∴ the solid angle at A = the solid angle at K. B. 11.

In the same manner,
the other solid angles of the figs are = one another.



Let then the solid fig. AG be applied to KQ: first, if the plane fig. AC be applied to the plane fig. KM, so that the | AB may coincide with KL;
the fig. AC must coincide with the fig. KM,
for they are equal and sim^r;
∴ the |^s AD, DC, CB coincide with KN, NM, ML,
each with each,
and the p^{ts} A, D, C, B, with the p^{ts} K, N, M, L:

B. 11 And the solid angle at A coincides with that at K;
∴ the plane AF coincides with the plane KP,
and the fig. AF with the fig. KP,
for they are equal and sim^r to one another:
∴ the |^s AE, EF, FB coincide with KO, OP, PL,
and the p^{ts} E, F with the p^{ts} O, P.

In the same manner,
the fig. AH coincides with the fig. KR,
and the |DH with NR, and the p^t H with R.

And,
∵ the solid angle at B = the solid angle at L,
∴ it may, in the same manner, be proved, that
the fig. BG coincides with the fig. LQ.
and the | CG with MQ, and the p^t G with Q.

Thus, all the planes and sides of the solid fig. AG coincide with the planes and sides of the solid fig. KQ,
each with each,
and ∴ AG is equal and sim^r to KQ.

And, in the same manner, it may be proved that *any other solid figures whatever contained by the same number of equal and similar planes, alike situated, and having none of their solid angles contained by more than three plane angles, are equal and similar to one another.* [Q. E. D.]

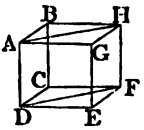
PROP. XXIV. THEOR.

If a solid be contained by six planes, two and two of which are parallel; the opposite planes are similar and equal parallelograms.

Let the solid fig. CDGH be contained by the $\parallel$ planes AC, GF; BG, CE; FB, AE: its opp. planes shall be sim^{r} and equal ▱^{s}. For,

∵ the two $\parallel$ planes BG, CE are cut by the plane AC,
∴ their com. sections AB, CD are $\parallel$: 16. 11.

again,
∵ the two $\parallel$ planes BF, AE are cut by the plane AC,
∴ their com. sections AD, BC are $\parallel$: 16. 11.
and, from above, AB, CD are $\parallel$;
∴ AC is a ▱.



In like manner, it may be proved that each of the fig^{s} CE, FG, GB, BF, AE is a ▱.

Join, AH, DF: then,
∵ AB is $\parallel$ DC, and BH $\parallel$ CF;
∴ the two $|^{s}$ AB, BH, w^{h} meet one another

are ∥ the two DC, CF, w^h meet one another and are not in the same plane with the other two

10.11. ∴ they contain equal ∠^s,

i.e. ∠ ABH = ∠ DCF:

And, ∵ AB, BH = DC, CF, each to each,

and ∠ ABH = ∠ DCF,

4.1. ∴ base AH = base DF,

and △ ABH = △ DCF:

34.1. but ▱ BG is double of △ ABH,

and ▱ CE is double of △ DCF:

∴ ▱ BG is equal and sim^r to ▱ EC.

In the same manner it may be proved that

▱ AC is equal and sim^r to ▱ GF,

and ▱ AE to BF.

∴ *if a solid, &c.* [Q. E. D.]

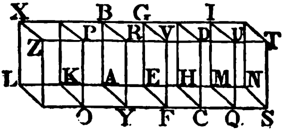
PROP. XXV. THEOR.

If a solid parallelopiped be cut by a plane parallel to two of its opposite planes; it divides the whole into two solids, the base of one of which shall be to the base of the other, as the one solid is to the the other.

Let the solid ⊞ ABCD be cut by the plane EV, w^h is ∥ the opp. planes, AR, HD, and div^s the whole into the two solids ABFV, EGCD: then sol. ABFV : sol. EGCD :: base AEFY : base EHCF

Prod. AH both ways, and take any n^o of |^s HM, MN, each = EH and any n^o AK, KL, each = EA;

and complete the $▱^s$ LO, KY, HQ, MS, and the solids LP, KR, HU, MT. Then,



∵ the $|^s$ LK, KA, AE are all equal,
∴ the $▱^s$ LO, KY, AF are equal; 36. 1.
and likewise the $▱^s$ KX, KB, AG: 24. 11.

also,
the $▱^s$ LZ, KP, AR, being opp. planes, are equal;

for the same reason,
the $▱^s$ EC, HQ, MS are equal, 36. 1.
and the $▱^s$ HG, HI, IN;
as also HD, MU, NT: 24. 11.
∴ three planes of the solid LP are = and sim^r to three planes of the solid KR,
as also to three planes of the solid AV;
but the three planes opp. to these three are = and sim^r to them in the several solids, and none of their 24. 11.
sol. angles are contained by more than three plane $\angle^s$;
∴ the three solids LP, KR, AV are = one another: C. 11.

for the same reason,
the three solids ED, HU, MT are = one another:
∴ whatever mult. the base LF is of the base AF,
the same mult. is the solid LV of the solid AV;
and whatever mult. the base NF is of the base HF,
the same mult. is the solid NV of the solid ED:
and as the base LF is >, = or < the base NF,
so the solid LV is >, = or < the solid NV. C. 11.

Hence,

∵ there are four magn^s, viz.

the two bases AF, FH, and the two solids AV, ED;

and that of the base AF and solid AV,

the base LF and solid LV are any equimult^s whatever;

and of the base FH and solid ED,

the base FN and solid NV are any equimult^s whatever;

and ∵ it has also been proved that

as the base LF is $>, =$ or $<$ the base NF,

so the solid LV is $>, =$ or $<$ the solid NV;

Def.5.5. ∴ solid AV : solid ED :: base AF : base FH.

∴ *if a solid, &c.* [Q. E. D.]

PROP. XXVI. PROB.

At a given point in a given straight line, to make a solid angle equal to a given solid angle contained by three plane angles.

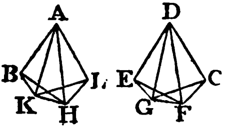
Let AB be a given |, A a given p^t in it, and D a given solid angle contained by the three plane $\angle^s$ EDC, EDF, FDC: it is req^d to make at the p^t A in the | AB a solid angle = the solid angle D.

11. 11. In the | DF take any p^t F, from w^h draw FG $\perp$ to the plane EDC, meeting that plane in G, and join DG: at the p^t A, in the | AB, make

23. 1. $\angle$ BAL $=$ $\angle$ EDC, and in the plane BAL make $\angle$ BAK $=$ $\angle$ EDG; then take AK $=$ DG, from

12. 11. the p^t K, erect KH at r^t $\angle^s$ to the plane BAL, make KH $=$ GF, and join AH. The solid angle at A

w^h is contained by the three plane $\angle^s$ BAL, BAH, HAL, shall be $=$ the solid angle at D contained by the three plane $\angle^s$ EDC, EDF, FDC.



Take the equal $|^s$ AB, DE, and join HB, KB, FE, GE: then,

$\because$ FG is $\perp$ to the plane EDC,

$\therefore$ it makes $r^t \angle^s$ with every $|$ meeting it in that plane: Def. 3. 11.

$\therefore$ each of the $\angle^s$ FGD, FGE is a $r^t \angle$.

For the same reason,

each of the $\angle^s$ HKA, HKB is a $r^t \angle$.

And,

$\because$ KA, AB $=$ GD, DE, each to each,

and that they contain equal $\angle^s$,

$\therefore$ the base BK $=$ the base EG; 4. 1.

and KH $=$ GF, Constr.

and HKB, FGE are $r^t \angle^s$,

$\therefore$ HB $=$ FE. 4. 1.

Again,

$\because$ AK, KH $=$ DG, GF, each to each,

and that they contain right $\angle^s$,

$\therefore$ the base AH $=$ the base DF:

and AB $=$ DE; Constr

$\therefore$ HA, AB $=$ FD, DE, each to each

and the base HB $=$ the base FE;

$\therefore$ $\angle$ BAH $=$ $\angle$ EDF. 8. 1.

For the same reason,

$\angle$ HAL = FDC :

for, making AL = DC, and joining KL, HL, GC, FC,

$\because$ the whole $\angle$ BAL = the whole $\angle$ EDC,

and, by the constr^n, the parts BAK, EDG are equal:

$\therefore$ the rem^g $\angle$ KAL = the rem^g $\angle$ GDC :

and $\because$ KA, AL = GD, DC, each to each,

and that they contain equal $\angle^s$,

4. 1. $\therefore$ the base KL = the base GC :

and KH = GF :

$\therefore$ KL, KH = GC, GF, each to each,

Def. 3. 11. and they also contain r^t $\angle^s$;

4. 1. $\therefore$ the base HL = the base FC :

Again, $\because$ HA, AL = FD, DC, each to each,

8. 1. and the base HL = the base FC,

$\therefore$ $\angle$ HAL = $\angle$ FDC,

Hence,

$\because$ the three plane $\angle^s$ BAL, BAH, HAL,

w^h contain the solid angle at A,

= the three plane $\angle^s$ EDC, EDF, FDC,

w^h contain the solid angle at D,

each to each, and are situated in the same order,

B. 11. $\therefore$ the solid angle at A = the solid angle at D.

$\therefore$ *at a given point in a given straight line has been made a solid angle equal to a given solid angle contained by three plane angles.* [Q. E. F.]

PROP. XXVII. THEOR.

To describe from a given straight line a solid parallelopiped similar and similarly situated to one given.

Let AB be the given |, and CD the given solid ⌸. It is req[d] from AB to desc. a solid ⌸ sim[r] and sim[ly] situated to CD.

At the p[t] A of the given | AB make a solid angle = the solid angle at C, and let BAK, KAH, BAH, be the three plane angles w[h] contain it, so that $\angle$BAK=$\angle$ECG, KAH=GCF, and HAB=FCE: 26. 11.

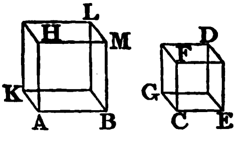
also make BA : AK :: EC : CG, 12. 6.

and AK : AH :: CG : CF, 12. 6.

whence, *ex æq.* BA : AH :: EC : CF: 22. 5.

then complete the ▱ BH, and the solid AL:

AL shall be sim[r] and sim[ly] situated to CD.



For, ∵ BA : AK :: EC : CG,

∴ the sides about the equal $\angle$[s] ECG, BAK are :: [ls],

and ∴ ▱ BK is sim[r] to ▱ EG, Def. 1. 6.

For the same reason,

▱ KH is sim[r] to GF, and HB to FE:

∴ three ▱[s] of the solid AL are sim[r] to three of the solid CD:

24. 11. and the three opp. ones in each solid are equal and sim^r to these, each to each.

Also,

∵ the plane ∠^{s} w^h contain the solid angles of the fig^s are equal, each to each, and situated in the same order,

B. 11. ∴ the solid angles are equal, each to each.

Def. 11. 11. ∴ the solid AL is sim^r to the solid CD.

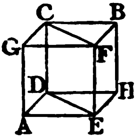
∴ *from a given straight line* AB *has been described a solid parallelopiped* AL *similar and similarly situated to the given one* CD.

[Q. E. F.]

PROP. XXVIII. THEOR.

If a solid parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes; it shall be cut into two equal parts.

Let AB be a solid ▱, and DE, CF the diagonals of the opp. ▱^{s} AH, GB, viz. those w^h are drawn betwixt the equal ∠^{s} in each:



∵ CD, FE are both ∥ GA, and not in the same plane with it,

9. 11. ∴ CD, FE are ∥;

∴ the diagonals CF, DE, are in

6. 11. the plane in w^h the ∥^{s} are, and are themselves ∥;

and the plane CDEF shall cut the solid AB into two equal parts.

For,

∴ △CGF = △CBF, and △DAE = △DHE, 34. 1.
and that the ▱ CA is equal and sim^r to the opp. one BE, and the ▱ GE to CH; 24. 11.
∴ the prism contained by the two △^{s} CGF, DAE, and the three ▱^{s} CA, GE, EC,
= the prism contained by the two △^{s} CBF, DHE, C. 11.
and the three ▱^{s} BE, CH, EC;
for they are contained by the same n^o of equal and sim^r planes, alike situated, and none of their solid angles are contained by more than three plane ∠^{s}.

∴ *the solid* AB *is bisected by the plane* CDEF.

[Q. E. D.]

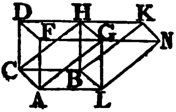
"N.B. The insisting straight lines of a parallelopiped mentioned in the next and some following propositions, are the sides of the parallelograms betwixt the base and the opposite plane parallel to it."

PROP. XXIX. THEOR.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the solid ⊞^{s} AH, AK be on the same base AB, and of the same altit., and let their insisting |^s AF, AG, LM, LN, be terminated in the same | FN, and CD, CE, BH, BK be terminated in the same | DK: the solid AH shall be = the solid AK.

First, let the ▱s DG, HN, wh are opp. to the base AB, have a com. side HG:
then,



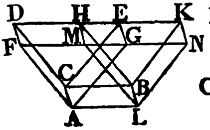
∵ the solid AH is cut by the plane AGHC passing through the diagonals AG, CH, of the opp. planes ALGF, CBHD,

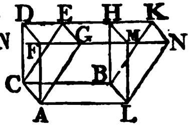
28. 11. ∴ AH is cut into two equal parts by the plane AGHC;
∴ the solid AH is double of the prism wh is contained betwixt the △s ALG, CBH:

for the same reason,
∵ the solid AK is cut by the plane LGHB, through the diagonals LG, BH, of the opp. planes ALNG, CBKH,
∴ the solid AK is double of the same prism wh is contained betwixt the △s ALG, CBH:

Ax. 6. ∴ the solid AH = the solid AK.

Next, let the ▱s DM, EN, opp. to the base, have no com. side: then,





∵ CH, CK are ▱s,

4. 1. ∴ CB = each of the opp. sides DH, EK;
∴ DH = EK:

add, or take away the com. part HE;

2 or 3 Ax. then DE = HK:

38. 1. ∴ also △ CDE = △ BHK,

36. 1. and ▱ DG = ▱ HN:

for the same reason,

△ AFG = △ LMN:

also ▱ CF = ▱ BM, and CG = BN, 24. 11.

for they are opposite;

∴ the prism wh is contained by the two △s AFG, CDE, and the three ▱s AD, DG, GC, = the prism contained by the two △s LMN, BHK, C. 11.

and the three ▱s BM, MK, KL.

If ∴ the prism LMN, BHK, be taken from the solid of wh the base is the ▱ AB, and in wh FDKN is the one opp. to it;

and if from this same solid there be taken the prism AFG, CDE;

the remg solid, viz. the ⊞ AH = the remg ⊞ AK. Ax. 3.

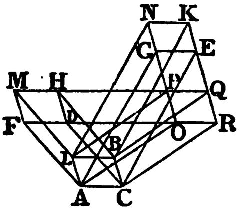
∴ *solid parallelopipeds, &c.* [Q. E. D.]

PROP. XXX. THEOR.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the ⊞s CM, CN, be on the same base AB, and of the same altit., but their insisting |s AF, AG, LM, LN, CD, CE, BH, BK, not terminated in the same |s: the solids CM, CN shall be equal to one another.

Prod. FD, MH, and NG, KE, and let them meet one another in the pts O, P, Q, R; and join AO, LP, BQ, CR: then,



∵ the plane LBHM is || the opp. plane ACDF, and that the plane LBHM is that in wh are the ||s LB, MHPQ, in wh also is the fig. BLPQ; and the plane ACDF is that in which are the ||s AC, FDOR, in wh also is the fig. CAOR;

∴ the figs BLPQ, CAOR are in || planes:

in like manner,

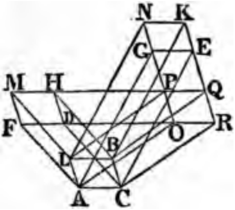
∵ the plane ALNG is || the opp. plane CBKE, and that the plane ALNG is that in wh are the ||s AL, OPGN, in wh also is the fig. ALPO; and the plane CBKE is that in wh are the ||s CB, RQEK, in wh also is the fig. CBQR;

∴ the figs ALPO, CBQR are in || planes:

Hyp. and the planes ACBL, ORQP are ||;

∴ the solid CP is a ▱:

29. 11. but the solid CM = the solid CP, for they are on the same base ACBL, and their insisting |s AF, AO, CD, CR; LM, LP, BH, BQ, are in the same |s FR, MQ.



and the solid CP = the solid CN, 29. 11.
for they are on the same base ACBL, and their insisting |s AO, AG, LP, LN ; CR, CE, BQ, BK are in the same |s ON, RK ;
∴ the solid CM = the solid CN.

∴ *solid parallelopipeds, &c.* [Q. E. D.]

PROP. XXXI. THEOR.

Solid parallelopipeds, which are upon equal bases, and of the same altitude, are equal to one another.

Let the solid ▱s AE, CF be on equal bases AB, CD, and be of the same altit.:
the solid AE shall be = the solid CF.

First, let the insisting |s be at rt ∠s to the bases AB, CD ; and let the bases be placed in the same plane, and so that the sides CL, LB may be in a | ; then the | LM, wh is at rt ∠s to the plane in wh the bases are, in the pt L, is com. to the two solids 13. 11.

AE, CF: let the other insisting |s of the solids be AG, HK, BE; DF, OP, CN: and first, let the
14. 1. ∠ ALB = the ∠ CLD: then AL, LD are in a |.*

Prod. OD, HB, and let them meet in Q, and complete the solid ▱ LR, the base of wh is the ▱ LQ, and of wh LM is one of its insisting |s.

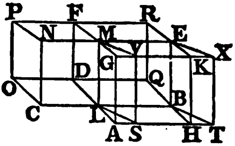
Then, ∵ ▱ AB = ▱ CD,

7. 5. ∴ base AB : LQ :: CD : LQ.

And,

∵ the solid ▱ AR is cut by the plane LMEB, wh is ∥ to the opp. planes AK, DR:

25. 11. ∴ solid AE : LR :: base AB : LQ:



for the same reason,

∵ the solid ▱ CR is cut by the plane LMFD, wh is ∥ to the opp. planes CP, BR;

∴ solid CF : LR :: base CD : LQ:

but, from above,

base AB : LQ :: CD : LQ;

11. 5. ∴ solid AE : LR :: CF : LR;

9. 5. and ∴ solid AE = solid CF.

But let the solid ▱s SE, CF be on equal bases SB, CD, and be of the same altit., and let their insisting |s be at rt ∠s to the bases; and place the bases SB, CD in the same plane, so that CL, LB

* See the note to Prop. 14. Book 6.

may be in a |; and let the ∠ˢ SLB, CLD be unequal: the solid SE shall be = the solid CF

Prod. DL, TS until they meet in A, and from B draw BH || DA; and let HB, OD prodᵈ meet in Q, and complete the solids AE, LR:

∴ the solid AE = the solid SE; 29. 11.

for they are on the same base LE, and of the same altit., and their insisting |ˢ, viz. LA, LS, BH, BT; MG, MV, EK, EX, are in the same |ˢ AT, GX:

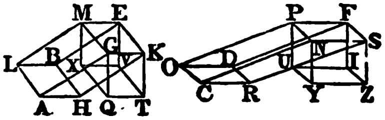
And, ∵ ▱ AB = SB, 35. 1.
for they are on the same base LB,
and between the same ||ˢ LB, AT;
and that the base SB = the base CD;
∴ the base AB = the base CD;
and ∠ ALB = ∠ CLD;

∴ by the first case,
the solid AE = the solid CF:

but it has been demᵖ that
the solid AE = the solid SE:
∴ the solid SE = the solid CF.

But, if the insisting |ˢ AG, HK, BE, LM; CN, RS, DF, OP be not at rᵗ ∠ˢ to the bases AB, CD; in this case also shall the solid AE = the solid CF.

From the ptˢ G, K, E, M; N, S, F, P, draw the |ˢ GQ, KT, EV, MX; NY, SZ, FI, PU, ⊥ to 11. 11.
the planes in wʰ are the bases AB, CD; and let them meet them in the ptˢ Q, T, V, X; Y, Z, I, U; and join QT, TV, VX, XQ; YZ, ZI, IU, UY.



Then,

∵ GQ, KT are at $r^t \angle^s$ to the same plane,

6. 11. ∴ they are || one another:

and MG, EK are $||^s$;

∴ the planes MQ, ET, of w^h one passes through MG, GQ, and the other through EK, KT, w^h are || MG, GQ, and not in the same plane with them,

15. 11. are || one another:

for the same reason,

the planes MV, GT are || one another:

∴ the solid QE is a ⊡.

In like manner, it may be proved that

the solid YF is a ⊡.

But, from what has been dem^d,

the solid EQ = the solid FY,

for they are on equal bases MK, PS, and of the some altit., and have their insisting $|^s$ at $r^t \angle^s$ to the bases:

29 or 30 11. and the solid EQ = the solid AE,

and the solid FY = the solid CF:

for they are on the same bases and of the same altit.;

∴ the solid AE = the solid CF.

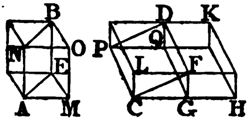
∴ *solid parallelopipeds, &c.* [Q. E. D.]

PROP. XXXII. THEOR.

Solid parallelopipeds which have the same altitude, are to one another as their bases.

Let AB, CD be solid ⊞s of the same altit.: they shall be to one another as their bases;

i. e. solid AB : solid CD :: base AE : base CF, Cor. 45.



To the | FG apply the ▱ FH = AE, so that ∠ FGH = ∠ LCG; and on the base FH complete the solid ⊞ GK, one of whose insisting |s is FD, whereby the solids CD, GK must be of the same altit.: then,

the solid AB = the solid GK, 31. 11.

for they are on equal bases AE, FH, and are of the same altit.;

And,

∵ the solid ⊞ CK is cut by the plane DG, wh is ∥ its opp. planes,

∴ base HF : FC :: solid HD : DC: 25. 11.

but, the base HF = the base AE,

and the solid GK = the solid AB;

∴ solid AB : CD :: base AE : CF.

∴ *solid parallelopipeds, &c.* [Q. E. D.]

Cor.—From this it is manifest, that prisms on triangular bases, of the same altit., are to one another as their bases.

Let the prisms, the bases of w^h are the $\triangle^s$ AEM, CFG, and NBO, PDQ the $\triangle^s$ opp. to them, have the same altit.: they shall be to one another as their bases.

Complete the ▱^{s} AE, CF; and the solid ⧈^{s} AB, CD, in the first of w^h let MO, and in the other let GQ be one of the insisting $|^s$. Then,

∵ the solid ⧈^{s} AB, CD, have the same altit.,

∴ they are to each other as the base AE is to the base CF:

28. 11 ∴ the prisms w^h are their halves are to each other as the base AE to the base CF,

i. e. as $\triangle$ AEM to $\triangle$ CFG.

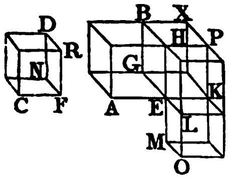
PROP. XXXIII. THEOR.

Similar solid parallelopipeds are one to another in the triplicate ratio of their homologous sides.

Let AB, CD be sim^r solid ⧈^{s}, and the side AE homol. to the side CF: the solid AB shall have to the solid CD the tripl. r^o of that w^h AE has to CF.

Prod. AE, GE, HE, and in these prod^d take EK = CF, EL = FN, and EM = FR; and complete the ▱ KL, and the solid KO. Then,

∵ KE, EL = CF, FN, each to each,
and ∠ KEL = ∠ CFN,
(for ∠ KEL = ∠ AEG,
and, since the solids AB, CD are sim^r,
∠ AEG = ∠ CFN);
∴ ▱ KL is sim^r and equal to ▱ CN.
For the same reason,
▱ MK is sim^r and equal to CR,
and also OE to FD:
∴ three ▱^s of the solid KO are sim^r and equal
to three ▱^s of the solid CD:
and the three opp. ones in each solid are sim^r and equal to these: 24. 11.
∴ the solid KO is sim^r and equal to the solid CD C. 11.



Complete the ▱ GK; and on the bases GK, KL, complete the solids EX, LP, so that EH be an insisting | in each of them, whereby they must be of the same altit. with the solid AB. Then,

∵ the solids AB, CD are sim^r,
and, by permutation,
AE : CF :: EG : FN
:: EH : FR;
but FC = EK, FN = EL, and FR = EM;
∴ AE : EK :: EG : EL
:: EH : EM:

1. 6. but ▱AG : ▱GK :: AE : EK;
1. 6. and ▱GK : ▱KL :: EG : EL;
also ▱PE : ▱KM :: EH : EM:
∴ ▱AG : ▱GK :: GK : KL
:: PE : KM:
25. 11. but solid AB : solid EX :: AG : GK;
25. 11. and solid EX : solid PL :: GK : KL;
25. 11. also solid PL : solid KO :: PE : KM:
∴ solid AB : solid EX :: EX : PL
:: PL : KO:

but if four magn$^{\text{s}}$ be continual :: $^{\text{ls}}$, the first is said
Def. 11. 5. to have to the fourth the tripl. r$^{\text{o}}$ of that w$^{\text{h}}$ it has to the second;
and ∴ the solid AB has to the solid KO, the tripl. r$^{\text{o}}$ of that w$^{\text{h}}$ AB has to EX:
but AB : EX :: ▱AG : ▱GK,
and :: | AE : | EK;
∴ the solid AB has to the solid KO, the tripl. r$^{\text{o}}$ of that w$^{\text{h}}$ AE has to EK:
but the solid KO = the solid CD,
and the | EK = the | CF;
∴ the solid AB has to the solid CD, the tripl. r$^{\text{o}}$ of that w$^{\text{h}}$ the side AE has to the homol. side CF.

∴ *similar solid parallelopipeds, &c.*

[Q. E. D.]

Cor.—From this it is manifest, that if four |$^{\text{s}}$ be continual :: $^{\text{ls}}$, as the first is to the fourth, so is the solid ▣ desc$^{\text{d}}$ from the first to the sim$^{\text{r}}$ solid sim$^{\text{ly}}$ desc$^{\text{d}}$ from the second; for the first | has to the fourth the trip. r$^{\text{o}}$ of that w$^{\text{h}}$ it has to the second.

PROP. D THEOR.

Solid parallelopipeds which are contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.

Let AB, CD be solid ⧉^{s}, of w^{h} AB is contained by the ▱^{s} AE, AF, AG, w^{h} are equiang^{r}, each to each, to the ▱^{s} CH, CK, CL, w^{h} contain the solid CD. The r^{o} w^{h} the solid AB has to the solid CD shall be the same with that w^{h} is compounded of the r^{os} of the sides AM to DL, AN to DK, and AO to DH.

Prod. MA, NA, OA to P, Q, R, so that AP = DL, AQ = DK, and AR = DH; and complete the solid ⧉ AX contained by the ▱^{s} AS, AT, AV sim^{r} and equal to CH, CK, CL, each to each: whence the solid AX = the solid CD. C. 11.

Complete likewise the solid AY, the base of w^{h} is AS, and AO one of its insisting |^{s}

Take any | a, and make 12. 6.

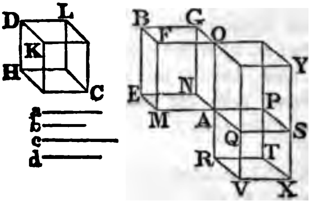
$$a : b :: MA : AP$$
$$b : c :: NA : AQ$$
$$c : d :: AO : AR.$$

Then, ∵ ▱ AE is equiang^{r} to AS,

∴ AE : AS :: | a : c, 23. 6.

and the solids AB, AY, being betwixt the || planes BOY, EAS, are of the same altit.;

32. 11. ∴ solid AB : solid AY :: base AE : base AS,
i. e. :: | a : c.



And,

33. 11. solid AY : solid AX :: base OQ : base QR,
i. e. :: | OA : AR
i. e. :: | c : d.

And,
∵ solid AB : solid AY :: a : c
and solid AY : solid AX :: c : d
∴ *ex æq.*
solid AB : AX or CD :: a : d.

Def. A. 5. But the r° of a to d is said to be compounded of the r°s of a to b, b to c, and c to d, wh are the same with the r°s of the sides MA to AP, NA to AQ, and OA to AR, each to each:
and the sides AP, AQ, AR = the sides DL, DK, DH, each to each.

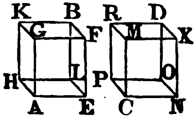
∴ *the solid* AB *has to the solid* CD *the ratio which is the same with that which is compounded of the ratios of the sides* AM *to* DL AN *to* DK, *and* AO *to* DH. [Q. E. D.]

PROP. XXXIV. THEOR.

The bases and altitudes of equal solid parallelopipeds, are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the solid parallelopipeds are equal.

Let AB, CD be two solid ▱s: and, first, let the insisting |s AG, EF, LB, HK; CM, NX, OD, PR, be at rt ∠s to the bases.

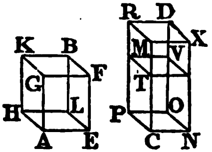
If the solid AB = the solid CD, their bases shall be reciprocally : :[1] to their altits;
i. e. base EH : base NP : : CM : AG.



If the base EH = the base NP; then,
∵ the solid AB is also = the solid CD,
∴ shall CM = AG :
for if the bases EH, NP be equal,
but the altits AG, CM be not equal,
neither shall the solid AB = the solid CD :
but these solids are equal, by hyps;
∴ the altit. CM is not ≠ AG;
i. e. CM = AG.
∴ base EH : base NP : : CM : AG.

Next, let the bases EH, NP not be equal, but EH > the other: then,
∵ the solid AB = the solid CD,
∴ CM > AG :

for, if it be not, neither also in this case would the solid AB = CD; whereas, by hyp^s, these solids are equal.



Make then CT = AG, and complete the solid ▱ CV, of w^h the base is NP, and altit. CT.

∵ the solid AB = the solid CD,

7. 5. ∴ the solid AB : CV :: CD : CV:

but,

32. 11. ∵ the solids AB, CV are of the same altit.,

25 11. ∴ the solid AB : CV :: the base EH : NP:

1. 6. and the solid CD : CV :: the base MP : PT;

and also :: | MC : CT;

and CT = AG:

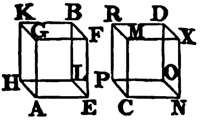
∴ the base EH : NP :: MC : AG:

∴ the bases of the solid ▱^s AB, CD are reciprocally ::[l] to their altit^s.

Let now the bases of the solid ▱^s AB, CD be reciprocally ::[l] to their altit^s, viz.

base EH : NP :: altit. CM : AG:

then shall the solid AB = the solid CD.



If the base EH = the base NP; then,

$\because$ EH : NP :: altit. of solid CD : altit. of solid AB,

$\therefore$ the altit. of CD = the altit. of AB: A. 5.

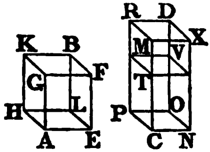
but solid ▱ˢ on equal bases, and of the same altit. are = one another: 31. 11.

$\therefore$ the solid CD = the solid AB.

But let the bases EH, NP be unequal; and let EH be the greater of the two: then,

$\because$ altit. CM of solid CD : altit. AG of solid AB, :: base EH : NP,

$\therefore$ CM is > AG. A. 5.



Hence, as before, take CT = AG, and complete the solid CV. Then,

$\because$ base EH : base NP :: CM : AG,

and that AG = CT,

$\therefore$ base EH : base NP :: CM : CT.

But, $\because$ the solids AB, CV are of the same altit.,

$\therefore$ solid AB : solid CV :: base EH : base NP: 32. 11.

also, MC : CT :: base MP : base PT, 1. 6.

:: solid CD : solid CV; 25. 11.

$\therefore$ solid AB : solid CV :: solid CD : solid CV,

i. e. each of the solids AB, CD has the same r° to the solid CV;

and $\therefore$ the solid AB = the solid CD. 9. 5.

Second General Case.—Let the insisting |ˢ FE, BL, GA, KH; XN, DO, MC, RP not be at rᵗ ∠ˢ to the bases of the solids.

In this case, likewise, if the solids AB, CD be equal, their bases shall be reciprocally $::^1$ to their altit^s, viz.
base EH : NP :: altit. of solid CD : altit. of AB.

From the p^{ts} F, B, K, G ; X, D, R, M draw $\perp^s$ to the planes in w^h are the bases EH, NP, meeting those planes in the p^{ts} S, Y, V, T ; Q, I, U, Z ; and complete the solids FV, XU, w^h are ▱^{s}, as was proved in the last part of Prop. 31 of this Book.

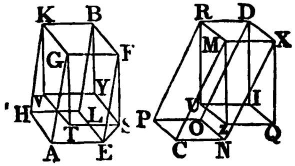
$\because$ the solid AB = the solid CD ;
and that the solids AB, BT, being on the same
29 or 30 11. base FK, and of the same altit., are equal ;
29 or 30 11. and that also the solids CD, DZ, being on the same base XR, and of the same altit. are equal ;
$\therefore$ the solid BT = the solid DZ :

but as was before proved, the bases are reciprocally $::^1$ to the altit^s of equal solid ▱^{s}, of w^h the insisting $|^s$ are at r^t $\angle^s$ to their bases ;
$\therefore$ altit. of solid DZ : altit. of BT :: base FK : XR;
and the base FK = EH, and the base XR = NP;
$\therefore$ altit. of solid DZ : altit. of BT :: base EH : NP;
but the altit^s of the solids DZ, DC, as also those of the solids BT, BA are the same :
$\therefore$ altit. of solid CD : altit. of AB :: base EH : NP;
i. e. the bases of the solid ▱^{s} AB, CD, are reciprocally $::^1$ to their altit^s.

Next, let the bases of the solids AB, CD be reciprocally $::^1$ to their altit^s, viz.
base EH : NP :: altit. of solid CD : altit. of AB;

then shall the solid AB = the solid CD.

For, the same constr^n being made,
∵ base EH : NP :: altit. of solid CD : altit. of AB,
and that the base EH = FK, and NP = XR;
∴ base FK : XR :: altit. of solid CD : altit. of AB:



but the altit^s of the solids AB, BT are the same,
as also those of CD and DZ;
∴ base FK : XR :: altit. of solid DZ : altit. of BT;
∴ the bases of the solids BT, DZ are reciprocally
:: [1] to their altit^s:
and their insisting |^s are at r^t ∠^s to the bases;
∴, as was before proved,
the solid BT = the solid DZ:
but BT = the solid BA, and DZ = the solid DC, 29 or 30 11.
for they are on the same bases, and of the same altit.;
∴ the solid AB = the solid CD.

∴ *the bases, &c.* [Q. E. D.]

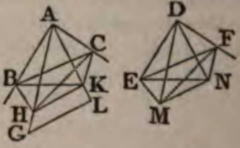
PROP. XXXV. THEOR.

If, from the vertices of two equal plane angles, there be drawn two straight lines elevated above the planes in which the angles are, and containing

equal angles with the sides of those angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the first named angles are; and from the points in which they meet the planes, straight lines be drawn to the vertices of the angles first named: these straight lines shall contain equal angles with the straight lines which are above the planes of the angles.

Let BAC, EDF be two equal plane $\angle^s$: and from the p^{ts} A, D let the $|^s$ AG, DM be elevated above the planes of the $\angle^s$, making equal $\angle^s$ with their sides, each to each, viz. $\angle$ GAB = MDE, and $\angle$ GAC = MDF; and in AG, DM, let any p^{ts} G, M be taken, and from them let $\perp^s$ GL, MN
11. 11. be drawn to the planes BAC, EDF, meeting these planes in the p^{ts} L, N; and join LA, ND:

$\angle$ GAL shall be = $\angle$ MDN.



Make AH = DM, and through H draw HK || GL:

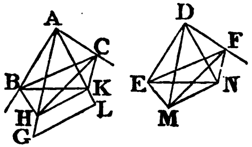
now GL is $\perp$ to the plane BAC,

8. 11. and ∴ HK is $\perp$ to the same plane.

From the p^{ts} K, N, to the $|^s$ AB, AC, DE, DF,

draw $\perp^s$ KB, KC, NE, NF;

and join HB, HC, ME, EF.



Then, ∵ HK is ⊥ to the plane BAC,

∴ the plane HBK, w^h passes through HK, 18. 11.

is at r^t ∠^{s} to the plane BAC;

and AB is drawn in the plane BAC at r^t ∠^{s} to the com. section BK of the two planes;

∴ AB is ⊥ to the plane HBK, and makes r^t ∠^{s} with every | meeting it in that plane; Def. 4. 11. Def. 3. 11.

but BH meets it in that plane;

∴ ABH is a r^t ∠:

for the same reason,

DEM is a r^t ∠;

and ∴ ∠ ABH = ∠ DEM:

and ∠ HAB = ∠ MDE: Hyp.

hence, in the two △^{s} HAB, MDE,

two ∠^{s} in the one = two ∠^{s} in the other, each to each

and one side HA = one side DM,

w^h sides are opp. to one of the equal ∠^{s} in each △;

∴ the rem^g sides are equal, each to each; 26. 1.

and ∴ AB = DE.

In the same manner, if HC and MF be joined, it may be dem^d that

AC = DF:

∴ AB, AC = DE, DF, each to each;

and ∠ BAC = ∠ EDF; Hyp.

∴ the base BC = the base EF, 4. 1.

and the rem^g ∠^{s} = the rem^g ∠^{s};

$\therefore \quad \angle ABC = \angle DEF$:

and the $r^t \angle ABK =$ the $r^t \angle DEN$;

$\therefore$ the $rem^g \angle CBK =$ the $rem^g \angle FEN$

for the same reason,

$\angle BCK = \angle EFN$:

Hence, in the two $\triangle^s$ BCK, EFN,

two $\angle^s$ in the one = two $\angle^s$ in the other, each to each;

and one side BC = one side EF,

w^h sides are adj^t to the equal $\angle^s$ in each $\triangle$;

$\therefore$ the other sides = the other sides;

$\therefore BK = EN$:

and $AB = DE$;

$\therefore AB, BK = DE, EN$, each to each;

and they contain $r^t \angle^s$;

$\therefore$ the base $AK =$ the base DN.

And, $\because AH = DM$,

$\therefore AH^2 = DM^2$;

but, $\because$ AKH and DNM are $r^t \angle^s$;

47. 1. $\therefore AH^2 = AK^2 + KH^2$,

and $DM^2 = DN^2 + NM^2$:

$\therefore \quad AK^2 + KH^2 = DN^2 + NM^2$:

and $AK^2 = DN^2$:

$\therefore$ the rem^g $KH^2 =$ the rem^g NM^2,

and $KH = NM$;

8. 1. and, $\because HA, AK = MD, DN$, each to each

and, from above,

base $HK = MN$;

$\therefore \angle HAK = \angle MDN$.

$\therefore$ *if from the vertices, &c.* [Q. E. D.]

COR.—From this it is manifest, that if from the vertices of two equal plane $\angle^s$, there be elevated

two equal |s containing equal ∠s with the sides of the ∠s, each to each; the ⊥s drawn from the extremities of the equal |s to the planes of the first ∠s are = one another.

Another Demonstration of the Corollary.

Let the plane ∠s BAC, EDF be = one another, and let AH, DM be two equal |s above the planes of the ∠s, containing equal ∠s with BA, AC, ED, DF, each to each, viz.

∠ HAB = MDE, and HAC = MDF;

and from H, M, let HK, MN be ⊥s to the planes BAC, EDF: HK shall be = MN.

For,

∵ the solid angle at A is contained by the three plane ∠s BAC, BAH, HAC, wh are, each to each, = the three plane ∠s EDF, EDM, MDF, containing the solid angle at D;

∴ the solid angles at A and D are equal,

and ∴ coincide with one another; to wit,

if the plane ∠ BAC be applied to the plane ∠ EDF,

the | AH coincides with DM,

as was shown in Prop. B of this Book:

and ∵ AH = DM,

∴ the pt H coincides with the pt M:

∴ HK, wh is ⊥ to the plane BAC,

coincides with MN, wh is ⊥ to the plane EDF, 13. 11.

for these planes coincide with one another,

∴ HK = MN. [Q. E. D.]

PROP. XXXVI. THEOR.

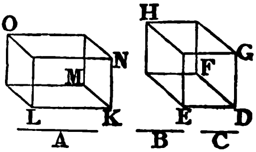
If three straight lines be proportionals, the solid parallelopiped described from all three, as its sides, is equal to the equilateral parallelopiped described from the mean proportional, one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the other figure.

Let A, B, C be three ∷ls, viz.

A : B ∷ B : C :

the solid descd from A, B, C shall be = the equilatl solid descd from B, equiangr to the other.

Take a solid angle D contained by three plane



∠s EDF, FDG, GDE: and make each of the |s ED, DF, DG = B, and complete the solid ▱ DH:
26. 11. make LK = A, and at the pt K in the | LK make a solid angle contained by the three plane ∠s LKM, MKN, NKL = the three ∠s EDF, FDG, GDE, each to each, and make KN = B, KM = C: and complete the solid ▱ KO.

Then, ∵ A : B :: B : C, and that
A = LK, B = each of the |s DE, DF, and C = KM;
∴ LK : DE :: DF : KM;
i.e. the sides about the equal ∠s are reciprocally :: l:
∴ ▱ LM = EF: 14. 6.

and,
∵ EDF, LKM are two equal plane ∠s,
and the two equal |s DG, KN are drawn from their vertices above their planes and contain equal ∠s with their sides;
∴ the ⊥s from the pts G, N, to the planes EDF LKM are = one another: Cor. 35. 11.
∴ the solids KO, DH are of the same altit.;
and they are on equal bases LM, EF;
and ∴ they are = one another: 31. 11.
but the solid KO is descd from the three |s A, B, C,
and the solid DH from the | B.

∴ *if three straight lines, &c.* [Q. E. D.]

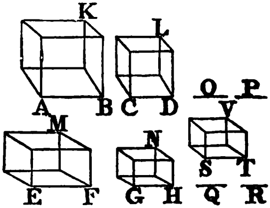
PROP. XXXVII. THEOR.

If four straight lines be proportionals, the similar solid parallelopipeds similarly described from them shall also be proportionals. And if the similar parallelopipeds similarly described from four straight lines be proportionals, the straight lines shall be proportionals.

Let the four |s AB, CD, EF, GH be :: ls, viz.
AB : CD :: EF : GH;

and let the sim[r] ◫[s] AK, CL, EM, GN be sim[ly] desc[d] from them: then shall

AK : CL :: EM : GN.



Make AB, CD, O, P continual ::[ls], as also EF, GH, Q, R: then,

	∵	AB :	CD	:: EF	: GH;
	and that	CD :	O	:: GH	: Q,
11. 5.	and	O :	P	:: Q	: R;
22.	∴ *ex æq.*	AB :	P	:: EF	: R:
Cor. 33. 11.	but solid	AK :	solid CL	:: AB	: P;
Cor. 33. 11.	and solid	EM :	solid GN	:: EF	: R;
11. 5.	∴ solid	AK :	solid CL	:: EM	: GN.

Next, let .

solid AK : solid CL :: solid EM : solid GN:

then shall | AB : CD :: EF : GH.

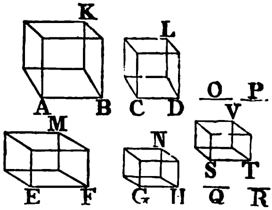
Take EF : ST :: AB : CD,

27. 11. and from ST desc. a solid ◫ SV sim[r] and sim[ly] situated to either of the solids EM, GN.

Then, ∵ AB : CD :: EF : ST,

and that from AB, CD the solid ◫[s] AK, CL are sim[ly] desc[d]; and, in like manner, the solids EM, SV from the |[s] EF, ST;

∴ AK : CL :: EM : SV;



but by hyp., AK : CL :: EM : GN ;

∴ GN = SV : 9. 5.

but it is likewise sim^r and siml^y situated to SV ; ∴ the planes w^h contain the solids GN, SV are sim^r and equal, and their homol. sides GH, ST are = one another :

and, ∵ AB : CD :: EF : ST,

and that ST = GH,

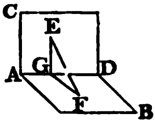
∴ AB : CD :: EF : GH.

∴ *if four straight lines, &c* [Q. E. D.]

PROP. XXXVIII. THEOR.

If a plane be perpendicular to another plane, and a straight line be drawn from a point in one of the planes perpendicular to the other plane, this straight line shall fall on the common section of the planes.

Let the plane CD be ⊥ to the plane AB, and let AD be the com. section : if any p^t E be taken in the plane CD, the ⊥ drawn from E to the plane AB shall fall on AD.



For, if it does not, let it, if possible, fall elsewhere, as EF; and let it meet the plane AB in

12. 1. the p^t F; and from F draw, in the plane AB, a $\perp$ FG to DA,

Def. 4. 11. w^h is also $\perp$ to the plane CD; and join EG.

Then, $\because$ FG is $\perp$ to the plane CD,

and the | EG, w^h is in that plane, meets it,

$\therefore$ FGE is a r^t $\angle$:

Def. 3. 11. but EF is also at r^t $\angle^s$ to the plane AB;

and $\therefore$ EFG is a r^t $\angle$:

$\therefore$ two $\angle^s$ of the $\triangle$ EFG are together = two r^t $\angle^s$,

17. 1. w^h is impossible:

$\therefore$ the $\perp$ from the p^t E to the plane AB does not fall elsewhere than upon the | AD;

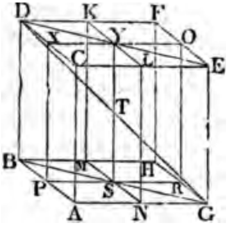
$\therefore$ it falls upon it.

$\therefore$ *if a plane, &c.* [Q. E. D.]

PROP. XXXIX. THEOR.

In a solid parallelopiped, if the sides of two of the opposite planes be divided, each into two equal parts, the common section of the planes passing through the points of division, and the diameter of the solid parallelopiped, cut each other into two equal parts.

Let the sides of the opp. planes CF, AH, of the solid ▱ AF be div^d each into two equal parts in the p^{ts} K, L, M, N; X, O, P, R; and join KL, MN, XO, PR: then,



∵ DK is = and ∥ CL,
∴ KL is ∥ DC: 33. 1.

for the same reason,
MN is ∥ BA:
and BA is ∥ DC;

hence,
∵ KL, BA are each of them ∥ DC,
and not in the same plane with it,
∴ KL is ∥ BA: 9. 11.

and ∵ KL, MN are each of them ∥ BA,
and not in the same plane with it,
∴ KL is ∥ MN: 9. 11
∴ KL, MN are in one plane.

In like manner it may be proved, that
XO, PR are in one plane.

Let YS be the com. section of the planes KN, XR; and DG the diam[r] of the solid ▱ AF: YS and DG shall meet, and cut one another into two equal parts.

Join DY, YE, BS, SG: then,
∵ DX is ∥ OE,
∴ the alt. ∠[s] DXY, YOE are equal: 29. 1.

hence, ∵ DX = OE, XY = YO,

4. 1. and that they contain equal ∠s;

∴ the base DY = the base YE,

and the other ∠s are equal;

∴ ∠ XYD = ∠ OYE,

14. 1. and ∴ DYE is a |:

for the same reason,

BSG is a |,

and BS = SG.

And

∵ CA is = and ∥ DB, and also = and ∥ EG;

9. 11. ∴ DB is = and ∥ EG:

and DE, BG join their extremities;

33. 1. ∴ DE is = and ∥ BG:

and DG, YS are drawn from pts in the one to pts in the other,

and ∴ are in one plane:

whence it is manifest that DG, YS must meet one another: let them meet in T. Then,

∵ DE is ∥ BG,

29. 1. ∴ the alt. ∠s EDT, BGT are equal:

15. 1. and ∠ DTY = ∠ GTS:

hence, in the △s DTY, GTS,

two ∠s in the one = two ∠s in the other, each to each,

and one side = one side, opp. to two of the equal ∠s

viz. DY = GS,

(for they are the halves of DE, BG),

26. 1. ∴ the rems sides are equal, each to each;

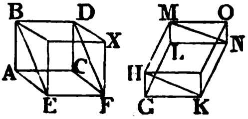
∴ DT = TG, and YT = TS.

∴ *if in a solid, &c* [Q. E. D.]

PROP. XL. THEOR.

If there be two triangular prisms of the same altitude, the base of one of which is a parallelogram, and the base of the other a triangle; if the parallelogram be double of the triangle, the prisms shall be equal to one another.

Let the prisms ABCDEF, GHKLMN be of the same altit., the first whereof is contained by the two △s ABE, CDF, and the three ▱s AD, DE, EC; and the other by the two △s GHK, LMN, and the three ▱s LH, HN, NG; and let one of them have a ▱ AF, and the other a △ GHK, for its base: if the ▱ AF be double of the △ GHK, then shall the prism ABCDEF = the prism GHKLMN.



Complete the solids AX, GO: then,
∵ ▱ AF is double of △ GHK,
and ▱ HK double of the same △; 31. 1.
∴ ▱ AF = ▱ HK:
but solid ⊞s on equal bases, and of the 31. 11.
same altit., are = one another;
∴ the solid AX = the solid GO:
and the prism ABCDEF is half of the solid AX; 28. 11.
and the prism GHKLMN half of the solid GO; 28. 11.
∴ the prism ABCDEF = the prism GHKLMN.

∴ *if there be two, &c.* [Q. E. D.]

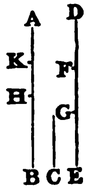
BOOK XII.

LEMMA I.

Which is the first proposition of the tenth book, and is necessary to some of the propositions of this book.

If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half; and so on: there shall at length remain a magnitude less than the least of the proposed magnitudes.

Let AB and C be two unequal magn[s] of w[h] AB is the greater. If from AB there be taken more than its half, and from the rem[r] more than its half, and so on; there shall at length remain a magn. $<$ C.



For C may be multiplied so as at length to become $>$ AB.

Let it be so multiplied: and let DE its mult. be $>$ AB, and let DE be div[d] into DF, FG, GE, each $=$ C.

From AB take BH $>$ its half, and from the rem[r] AH take HK $>$ its half, and so on; until there are as many divisions in AB as there are in DE; and

let the divisions in AB be AK, KH, HB; and the divisions in DE be DF, FG, GE.

Then, $\because$ DE is $>$ AB,
and that EG taken from DE is $\ngtr$ its half,
but BH taken from AB is $>$ its half;
$\therefore$ the rem^r GD is $>$ the rem^r HA.

Again, $\because$ GD is $>$ HA,
and that GF is $\ngtr$ the half of GD,
but HK is $>$ the half of HA;
$\therefore$ the rem^r FD is $>$ the rem^r AK.

And, FD $=$ C;
$\therefore$ C is $>$ AK;
i.e. AK is $<$ C. [Q. E. D.]

And if only the halves be taken away, the same thing may in the same way be demonstrated.

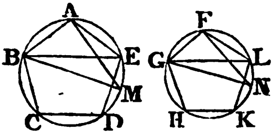
PROP. I. THEOR.

Similar polygons inscribed in circles, are to one another as the squares of their diameters.

Let ABCDE, FGHKL be two $\odot^s$, and in them the sim^r polygons ABCDE, FGHKL; and let BM, GN, be the diam^{rs} of the $\odot^s$:

polyg^{n}ABCDE : polyg^{n}FGHKL :: BM^2 : GN^2.

Join BE, AM, GL, FN: then,
$\because$ polygon ABCDE is sim^r to polygon FGHKL;
$\therefore \angle$ BAE $= \angle$ GFL,
and BA : AE :: GF : FL:



hence, $\because$ in the two $\triangle^s$ BAE, GFL,
one $\angle$ = one $\angle$,
and the sides about the equal $\angle^s$ are $::^{ls}$,
$\therefore$ the $\triangle^s$ are equiang^r;
and $\therefore \angle AEB = \angle FLG$:

but,
$\because \angle^s$AEB, AMB stand on the same part of the $\odot^{ce}$;
21. 3. $\therefore \angle AEB = \angle AMB$;

and for the same reason,
$\angle FLG = \angle FNG$:
$\therefore$ also $\angle AMB = \angle FNG$:
31. 3. and r$^t \angle BAM =$ r$^t \angle GFN$;
$\therefore$ the rem^g $\angle^s$ in the $\triangle^s$ABM, FGN are equal,
and the $\triangle^s$ are equiang^r to one another:
4. 6. $\therefore BA : GF :: BM : GN$;
10 Def 5 & 22. 5. and $\therefore$ the dupl. r^o of BM to GN is the same with the dupl. r^o of BA to GF:
but the r^o of BM^2 to GN^2 is the dupl. r^o of that
20. 6. w^h BM has to GN;
and the r^o of the polygon ABCDE to the polygon
20. 6. FGHKL is the dupl. of that w^h BA has to GF:
$\therefore$pol^n ABCDE : pol^n FGHKL :: $BM^2 : GN^2$.

$\therefore$ *similar polygons, &c.* [Q. E. D.]

PROP. II. THEOR.

Circles are to one another as the squares of their diameters.

Let ABCD, EFGH be two ⊙s; BD, FH their diamrs:

⊙ ABCD : ⊙ EFGH :: $BD^2 : FH^2$.

For, if it be not so, then must

$BD^2 : FH^2$ :: ⊙ ABCD : { some space either > or < ⊙ EFGH*

First, let this space be a space S < ⊙ EFGH; and in the ⊙ EFGH desc. the sq. EFGH. 6. 4.

This sq. is > half of the ⊙ EFGH;
for if, through the pts E, F, G, H there be drawn tangents to the ⊙,
the sq. EFGH is half of the sq. descd about the ⊙: 41. 1.
and the ⊙ is < the sq. descd about it:
∴ the sq. EFGH is > half of the ⊙.

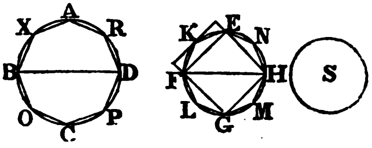
Bist each of the arcs EF, FG, GH, HE, in the ptsK, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE:

* For there is some sq. = ⊙ ABCD;
let P be the side of it;
and to three |s BD, FH and P there can be a fourth ::l:
let this be Q;
the sqs of these four |s are ::ls;
i. e. it is possible that to the sqs of BD, FH, 22. 6.
and the ⊙ ABCD there may be a fourth ::l.
Let this be S.

And in like manner are to be understood some things in the following propositions.

then each of the △s EKF, FLG, GMH, HNE
is > half the seg^t of the ⊙ in w^h it stands;
for, if |s touching the ⊙ be drawn through the p^ts
K, L, M, N, and the ▱s on the |s EF, FG, GH, HE
be completed,
each of the △s EKF, FLG, GMH, HNE is the
41. 1. half of the ▱ in w^h it is:

but every seg^t is < the ▱ in which it is;
∴ each of the △s EKF, FLG, GMH, HNE
is > half the seg^t of the ⊙ w^h contains it.



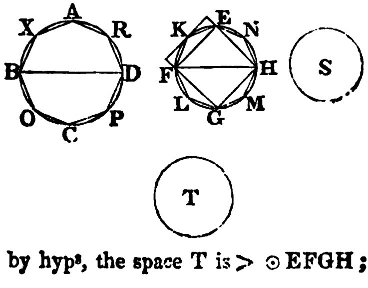
Again, if the rem^g arcs be each of them bis^d, and their extremities be joined by |s, by continuing to do this, there will at length remain seg^ts of the ⊙, w^h together are < the excess of the ⊙ EFGH above the space S; for, by the preceding Lemma, if from the greater of two unequal magn^s there be taken more than its half, and from the rem^r more than its half, and so on; there shall at length remain a magn. < the least of the proposed magn^s.

Let then the seg^ts EK, KF, FL, LG, GM, MH, HN, NE be those that remain, and are together
< the excess of the ⊙ EFGH above S:
∴ the rest of the ⊙, viz. the polyg^n EKFLGHMN
is > the space S.

Desc. likewise in the $\odot$ ABCD the polygon AXBOCPDR sim[r] to the polygon EKFLGMHN,
$\therefore$ polyg[n] AXBOCPDR : polyg[n] EKFLGMHN
:: $BD^2 : FH^2$. 1. 12.
but also, $BD^2 : FH^2$ :: $\odot$ ABCD : space S : Hyp.
$\therefore$ polyg[n] AXBOCPDR : polyg[n] EKFLGHMN 11. 5.
:: $\odot$ ABCD : space S.
but the $\odot$ ABCD is $>$ the polygon contained in it;
$\therefore$ the space S is $>$ the polyg[n] EKFLGHMN:
but, as has been dem[d], 14. 5.
the space S is also $<$ the above polygon;
w[h] is impossible.
$\therefore$ it is impossible that
$BD^2 : FH^2$ :: $\odot$ ABCD : any space $<$ $\odot$ EFGH.

In the same manner it may be dem[d] to be impossible that
$FH^2 : BD^2$ :: $\odot$ EFG : any space $<$ $\odot$ ABCD;

Neither is it possible that
$BD^2 : FH^2$ :: $\odot$ ABCD : any space $>$ $\odot$ EFGH.
For, if possible, let this space be T, then inv[ly];
$FH^2 : BD^2$ :: space T : $\odot$ ABCD:



but, by hyp[s], the space T is $>$ $\odot$ EFGH; and

$\therefore$ space T : ⊙ ABCD

14. 5. :: ⊙ EFGH : some space* < ⊙ ABCD;

$\therefore$ FH^2 : BD^2

:: ⊙ EFGH : some space < ⊙ ABCD

w[h] has been shown to be impossible:

$\therefore$ it is impossible that

BD^2 : FH^2 :: ⊙ ABCD : any space > ⊙ EFGH:

and it has also been dem[d] that it is impossible that

BD^2 : FH^2 :: ⊙ ABCD : any space < ⊙ EFGH

$\therefore$ BD^2 : FH^2 :: ⊙ ABCD : ⊙ EFGH : †

$\therefore$ *circles are, &c.* [Q. E. D.]

PROP. III. THEOR.

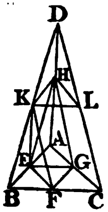
Every pyramid having a triangular base, may be divided into two equal and similar pyramids having triangular bases, and which are similar to the whole pyramid; and into two equal prisms which together are greater than the half of the whole pyramid.

Let there be a pyr[d] of w[h] the base is the △ ABC

* For as, in the foregoing note it was explained how it was possible there could be a fourth ⁝⁝[l] to the squares of BD, FH, and the ⊙ ABCD, w[h] was named S; so, in like manner, there can be a fourth ⁝⁝[l] to this other space, named T, and the ⊙[s] ABCD, EFGH. And the like is to be understood in some of the following propositions.

† For, as a fourth ⁝⁝l to the sq[s] of BD, FH,
and the ⊙ ABCD is possible,
and that it can neither be > nor < ⊙ EFGH,
$\therefore$ it must be = it.

and its vertex the p^t D: the pyr^d ABDC may be div^d into two equal and sim^r pyr^{ds} having triangular bases, and sim^r to the whole; and into two equal prisms w^h together shall be > half of the whole pyramid.



Bis^t AB, BC, CA, AD, DB, DC in the p^{ts} E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG: then,

∵ $AE = EB$, and $AH = HD$,
∴ HE is ∥ to DB: 2. 6.

for the same reason,

HK is ∥ to AB:
∴ HEBK is a ▱, 34. 1.
and $HK = EB$:
but $EB = AE$;
∴ also $AE = HK$;
and $AH = HD$: Constr.
∴ EA, AH = KH, HD each to each;
and ∠ EAH = ∠ KHD; 29. 1.
∴ the base EH = the base KD,
and △ AEH is equal and sim^r to △ HKD. 4. 1.

For the same reason,

△ AGH is equal and sim^r to △ HLD.

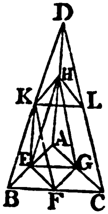
Again,

∵ the two $|^s$ EH, HG, w^h meet one another,
are ∥ to KD, DL, w^h meet one another,
and are not in the same plane with them
∴ they contain equal $∠^s$; 10. 11.
∴ ∠ EHG = ∠ KDL:

Hence, in the $\triangle^s$ EHG, KDL,
$\because$ EH, HG $=$ KD, DL, each to each,
and $\angle$ EHG $=$ $\angle$ KDL ;
$\therefore$ the base EG $=$ the base KL,
4. 1. and $\triangle$ EHG is equal and sim^r to $\triangle$ KDL.

For the same reason,
$\triangle$ AEG is equal and sim^r to $\triangle$ HKL.
$\therefore$ the pyr^d, of which the base is the $\triangle$ AEG, and
C. 11. of w^h the vertex is the p^t H, is equal and sim^r to the pyr^d, the base of w^h is the $\triangle$ KHL, and vertex the p^t D.

And,
$\because$ HK is $\parallel$ to AB, a side of $\triangle$ ADB,
$\therefore$ $\triangle$ ADB is equiang. to $\triangle$ HDK,
4. 6. and their sides are $::^{ls}$:
$\therefore$ $\triangle$ ADB is sim^r to $\triangle$ HDK :
and for the same reason,
$\triangle$ DBC is sim^r to $\triangle$ DKL,
and $\triangle$ ADC to $\triangle$ HDL,
and also $\triangle$ ABC to $\triangle$ AEG ;



but, as was before proved,
21. 6. $\triangle$ AEG is sim^r to $\triangle$ HKL ;
$\therefore$ $\triangle$ ABC is sim^r to $\triangle$ HKL :
and $\therefore$ the pyr^d of which the base is the $\triangle$ ABC and vertex the p^t D, is sim^r to the pyr^d of w^h the base is the $\triangle$ HKL, and vertex the same p^t D :

but, as has been proved,
B. 11. & Def. 11. 11. the pyr^d of w^h the base is the $\triangle$ HKL, and vertex the p^t D, is sim^r to the pyr^d the base of w^h is the $\triangle$ AEG, and vertex the p^t H :

∴ the pyr^{d}, the base of w^{h} is the △ ABC, and vertex the p^{t} D, is sim^{r} to the pyr^{d} of w^{h} the base is
the △ AEG, and vertex H :

∴ each of the pyr^{ds} AEGH, HKLD is sim^{r}
to the whole pyr^{d} ABCD.

And, ∵ BF = FC,
∴ ▱ EBFG is double of △ GFC : 41. 1.
but when there are two prisms of the same altit.
of w^{h} one has a ▱ for its base,
and the other a △ that is half of the ▱,
these prisms are = one another ; 40. 11
∴ the prism having the ▱ EBFG for its base, and the | KH opp. to it, is = the prism having the △ GFC for its base, and the △ HKL opp. to it ;
for the prisms are between the || planes ABC, HKL, 15. 11.
and ∴ are of the same altit. :

and it is manifest that each of these prisms is > either of the pyr^{ds} of which the △^{s} AEG, HKL are the bases and the vertices the p^{ts} H, D ; for, if EF be joined, the prism having the ▱ EBFG for its base, and KH the | opp. to it, is > the pyr^{d} of w^{h} the base is the △ EBF, and vertex the p^{t} K :

but this pyr^{d} is = the pyr^{d}, the base of which is C. 11.
the △ AEG, and vertex the p^{t} H ;
for they are contained by equal and sim^{r} planes :
∴ the prism having the ▱ EBFG for its base, and
opp. side KH, is > the pyr^{d} of which the base is
the △ AEG, and vertex the p^{t} H :

and the prism of w^{h} the base is the ▱ EBFG, and opp. side KH, is = the prism having the △ GFC for its base, and HKL the △ opp. to it ;

and the pyr^d of w^h the base is the $\triangle$ AEG, and vertex H, is = the pyr^d of w^h the base is the $\triangle$ HKL and vertex D:

$\therefore$ the two prisms before-mentioned are $>$ the two pyr^{ds} of w^h the bases are the $\triangle^s$ AEG, HKL, and vertices the p^{ts} H, D.

$\therefore$ *the whole pyramid of which the base is the triangle* ABC, *and vertex the point* D, *is divided into two equal pyramids similar to one another, and to the whole pyramid; and into two equal prisms; and the two prisms are together greater than half of the whole pyramid.* [Q. E. D.]

PROP. IV. THEOR.

If there be two pyramids of the same altitude, upon triangular bases, and each of them be divided into two equal pyramids, similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on; as the base of one of the first two pyramids is to the base of the other, so shall all the prisms in one of them be to all the prisms in the other, that are produced by the same number of divisions.

Let there be two pyr^{ds} of the same altit. on the triangular bases ABC, DEF, and having their vertices in the p^{ts} G, H; and let each of them be div^d into two equal pyr^{ds} sim^r to the whole, and also into two equal prisms; and let each of the pyr^{ds} thus

made be conceived to be div^{d} in like manner, and so on:

as the base ABC is to the base DEF,
so shall all the prisms in the pyr^{d} ABCG
be to all the prisms in the pyr^{d} DEFH
made by the same n^{o} of divisions.

Construct as in the foregoing prop^{n}; then,

$\because$ BX = XC, and AL = LC;
$\therefore$ XL is $\parallel$ AB, 2. 6.
and $\triangle$ ABC sim^{r} to $\triangle$ LXC.

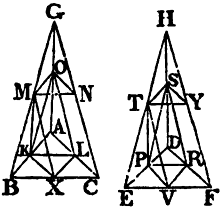
For the same reason,
$\triangle$ DEF is sim^{r} to $\triangle$ RVF.

And,
$\because$ BC is double of CX, and EF double of FV,
$\therefore$ BC : CX :: EF : FV: C. 5.
and on BC, CX are desc^{d} the sim^{r} and sim^{ly} situated rect^{l} fig^{s} ABC, LXC; and on EF, FV, in like manner, are desc^{d} the sim^{r} fig^{s} DEF, RVF;
$\therefore$ $\triangle$ ABC : $\triangle$ LXC :: $\triangle$ DEF : $\triangle$ RVF; 22. 6.
and, by permutation,
$\triangle$ ABC : $\triangle$ DEF :: $\triangle$ LXC : $\triangle$ RVF.

And,
$\because$ the planes ABC, OMN are $\parallel$,*
as also the planes DEF, STY, 15. 11.
$\therefore$ the $\perp$^{s} from the p^{ts} G, H, to the bases ABC, DEF,
w^{h}, by hyp., are = one another,
shall each be bis^{d} by the planes OMN, STY, 17. 11.

* $\because$ GO = OA, and GM = MB, 2. 6.
$\therefore$ OM is $\parallel$ to AB;
and in like manner,
ON is $\parallel$ to AC;
$\therefore$ the plane MON is $\parallel$ to the plane BAC. 15. 11.

17. 11. for the $|^{s}$ GC, HF are bis^{d} in the p^{ts} N, Y,
by the same planes:
∴ the prisms LXCOMN, RVFSTY
are of the same altit.;
and ∴ as the base LXC is to the base RVF,
Cor. 32. 11. *i. e.* as the $\triangle$ ABC is to the $\triangle$ DEF,
so is the prism having the $\triangle$ LXC for its
base, and OMN the $\triangle$ opp. to it,
to the prism of w^{h} the base is the $\triangle$ RVF,
and the opp. $\triangle$ STY:



and ∵ the two prisms in the pyr^{d} ABCG are equal
and also the two prisms in the pyr^{d} DEFH;
∴ as the prism of w^{h} the base is the ▱ KBXL
and opp. side MO, to the prism having the $\triangle$ LXC
for its base, and OMN the $\triangle$ opp. to it;
7. 5. so is the prism of w^{h} the base is the ▱ PEVR
and opp. side TS, to the prism of w^{h} the base
is the $\triangle$ RVF, and opp. $\triangle$ STY:
∴ by composition,
as the prisms KBXLMO, LXCOMN together
are to the prism LXCOMN,
so are the prisms PEVRTS, RVFSTY,
to the prism RVFSTY;

and, permutando,

as the prisms KBXLMO, LXCOMN,
are to the prisms PEVRTS, RVFSTY,
so is the prism LXCOMN to the prism RVFSTY;
but, as has been proved,
as the prism LXCOMN is to the prism RVFSTY,
so is the base ABC to the base DEF;
$\therefore$ as the base ABC to the base DEF,
so are the two prisms in the pyr[d] ABCG
to the two prisms in the pyr[d] DEFG:
and likewise if the pyr[ds] now made, for example
the two OMNG, STYH be similarly div[d],
as the base OMN is to the base STY,
so are the two prisms in the pyr[d] OMNG
to the two prisms in the pyr[d] STYH:

but,
base OMN : base STY :: hase ABC : base DEF:
$\therefore$ as the base ABC is to the base DEF,
so are the two prisms in the pyr[d] ABCG
to the two prisms in the pyr[d] DEFH;
and so are the two prisms in the pyr[d] OMNG
to the two prisms in the pyr[d] STYH:
and so are all four to all four:
and the same thing may be shown of the prisms made by dividing the pyr[ds] AKLO and DPRS, and of all made by the same n[o] of divisions.

[Q. E. D.]

PROP. V. THEOR.

Pyramids of the same altitude which have triangular bases, are to one-another as their bases.

Let the pyr[ds] of which the △[s] ABC, DEF are the bases, and of w[h] the vertices are the p[ts] G, H, be of the same altit.: then,
pyr[d] ABCG : pyr[d] DEFH : : base ABC : base DEF.

For, if it be not so, it follows that
base ABC : base DEF
: : pyr[d] ABCG : a solid either $<$ or $>$ pyr[d] DEFH.*

First, let this solid be a solid, Q, $<$ the pyr[d] : and div. the pyr[d] DEFH into two equal pyr[ds], sim[r] to the whole, and into two equal prisms ; then,
3. 12. these two prisms are $>$ the half of the whole pyr[d].

Again, let the pyr[ds] made by this division be
Lem. 1. 12. in like manner div[d], and so on until the pyr[ds] w[h] remain undiv[d] in the pyr[d] DEFH be, all of them together, $<$ the excess of the pyr[d] DEFH above the solid Q :

let these, for example, be the pyr[ds] DPRS, STYH :
∴ the prisms, w[h] make the rest of the pyr[d] DEFH, are $>$ the solid Q.

* This may be explained in the same way as at the note in Prop. 2., in the like case.

Let the pyr[d] ABCG be also div[d] in the same manner, and into as many parts, as the pyr[d] DEFH;

∴ the prisms in the pyr[d] ABCG
: the prisms in the pyr[d] DEFH
:: base ABC : base DEF: 4. 12.

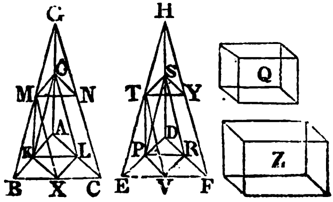
but, by hyp[s],
pyr[d] ABCG : solid Q :: base ABC : base DEF;
and ∴ the prisms in the pyr[d] ABCG
: the prisms in the pyr[d] DEFH
:: pyr[d] ABCG : solid Q:

but the pyr[d] ABCG is > the prisms contained in it,
∴ also the solid Q is > the prisms in the pyr[d] DEFH; 14. 5.

but it is also < those prisms,
w[h] is impossible.

∴ it is not the case that
base ABC : base DEF:
:: pyr[d] ABCG : any solid < pyr[d] DEFH.

In the same manner it may be dem[d] to be impossible that
base DEF : base ABC
:: pyr[d] DEFH : any solid < pyr[d] ABCG.



Nor is it possible that

base ABC : base DEF

:: pyr^d ABCG : any solid < pyr^d DEFH.

For, if it be possible, let this solid be Z : then,

∵ base ABC : base DEF :: pyr^d ABCG : solid Z

∴ by inv^n,

base DEF : base ABC :: solid Z : pyr^d ABCG;

but, ∵ solid Z is > the pyr^d DEFH,

∴ solid Z : pyr^d ABCG

4. 5. :: pyr^d DEFH : some solid* < pyr^d ABCG;

and ∴

base DEF : base ABC

:: pyr^d DEFH : some solid < pyr^d ABCG,

but the contrary to this has been proved:

∴ it is not the case that

base ABC : base DEF

:: pyr^d ABCG : any solid > pyr^d DEFH.

And it has been proved that neither is

base ABC : base DEF

:: pyr^d ABCG : any solid < pyr^d DEFH.

∴

base ABC : base DEF :: pyr^d ABCG : pyr^d DEFH.

∴ *pyramids, &c.* [Q. E. D.]

* This may be explained in the same way as at the like case in Prop. 2.

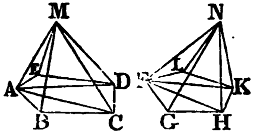
PROP. VI. THEOR.

Pyramids of the some altitude which have polygons for their bases, are to one another as their bases.

Let the pyr^{ds} w^{h} have the polygons ABCDE, FGHKL, for their bases, and their vertices in the p^{ts} M, N, be of the same altit.: then,

pyr^{d} ABCDEM : pyr^{d} FGHKLN
:: base ABCDE : base FGHKL.

Div. the base ABCDE into the △^{s} ABC, ACD, ADE, and the base FGHKL into the △^{s} FGH, FHK, FKL: and on the bases ABC, ACD, ADE, let there be as many pyr^{ds} of w^{h} the com. vertex is the p^{t} M, and on the rem^{g} bases as many pyr^{ds} having their com. vertex in the point N:



then, ∵ △ ABC : △ FGH
:: pyr^{d} ABCM : pyr^{d} FGHN; 5. 12.
and △ ACD : △ FGH
:: pyr^{d} ACDM : pyr^{d} FGHN;
and also △ ADE : △ FGH
:: pyr^{d} ADEM : pyr^{d} FGHN;

as all the first antecedents to their com. conseq^{t}. 2. Cor.
so are all the other antecedents to their com. conseq^{t}; 24. 5.

i. e. pyr^d ABCDEM : pyr^d FGHN
:: base ABCDE : base FGH:

and for the same reason,
pyr^d FHGKLN : pyr^d FGHN
:: base FGHKL : base FGH;
and, by inv^n,
pyr^d FGHN : pyr^d FHGKLN
:: base FGH : base FGHKL:

then, ∵ pyr^d ABCDEM : pyr^d FGHN
:: base ABCDE : base FGH:
and pyr^d FGHN : pyr^d FGHKLN
:: base FGH : base FGHKL;

∴ *ex æq.*
22. 5. pyr^d ABCDEM : pyr^d FGHKLN
:: base ABCDE : base FGHKL.

∴ *pyramids, &c.* [Q. E. D.]

PROP. VII. THEOR.

Every prism having a triangular base may be divided into three pyramids that have triangular bases, and are equal to one another.

Let there be a prism of which the base is the △ ABC, and DEF the △ opp. to it: the prism ABCDEF may be div^d into three equal pyr^{ds} having triangular bases.

Join BD, EC, CD: then,
∵ ABED is a ▱ of w^h BD is the diam^r.

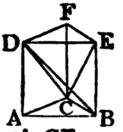
$\therefore \triangle ABD = \triangle EBD$; 34. 1.

$\therefore$ the pyr^d of w^h the base is the $\triangle$ ABD, and vertex the p^t C, is $=$ the pyr^d of w^h the base is the $\triangle$ EBD, and vertex the p^t C: 5. 12.

but this pyr^d is the same with the pyr^d the base of w^h is the $\triangle$ EBC, and vertex the p^t D;

for they are contained by the same planes:

$\therefore$ the pyr^d of w^h the base is the $\triangle$ ABD, and vertex the p^t C, is $=$ the pyr^d, the base of w^h is the $\triangle$EBC, and vertex the p^t D.



Again,

$\because$ FCBE is a $\square$, of w^h the diam^r is CE,

$\therefore \triangle ECF = \triangle ECB$:

$\therefore$ the pyr^d of w^h the base is the $\triangle$ ECB, and 34. 1.
vertex the p^t D, is $=$ the pyr^d the base of w^h is the $\triangle$ ECF, and vertex the p^t D:

but it has been proved that the pyr^d of w^h the base is the $\triangle$ ECB, and vertex the p^t D, is $=$ the pyr^d of w^h the base is the $\triangle$ ABD, and vertex the p^t C;

$\therefore$ the prism ABCDEF is div^d into three equal pyr^{ds} having triangular bases, viz.

into the pyr^{ds} ABDC, EBDC, ECFD.

And,

$\because$ the pyr^d of w^h the base is the $\triangle$ ABD, and vertex the p^t C, is the same with the pyr^d of w^h the base is the $\triangle$ ABC, and vertex the p^t D,

for they are contained by the same planes;

and that the pyr^d of w^h the base is the $\triangle$ ABD, and vertex the p^t C, has been dem^d to be a third part of the prism, the base of w^h is the $\triangle$ ABC, and DEF the opp. $\triangle$;

∴ the pyr^{d} of w^{h} the base is the △ ABC, and vertex the p^{t} D, is the third part of the prism w^{h} has the same base, viz. the △ ABC, and DEF its opp. △. [Q. E. D.]

COR. 1.—From this it is manifest, that every pyr^{d} is the third part of a prism w^{h} has the same base, and is of an equal altit. with it: for if the base of the prism be any other fig. than a △, it may be div^{d} into prisms having triangular bases.

COR. 2.—Prisms of equal altit^{s} are to one another as their bases; for the pyr^{ds} on the same bases, and
5. 12 of the same altit., are to one another as their bases.

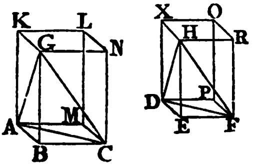
PROP. VIII. THEOR.

Similar pyramids, having triangular bases, are one to another in the triplicate ratio of that of their homologous sides.

Let the pyr^{ds} having the △^{s} ABC, DEF, for tneir bases, and the p^{ts} G, H, for their vertices, be sim^{r} and sim^{ly} situated: the pyr^{d} ABCG shall have to the pyr^{d} DEFH the tripl. r^{o} of that w^{h} the side BC has to the homol. side EF.

Complete the ▱^{s} ABCM, GBCN, ABGK, and the solid ▱ BGML, contained by these planes and those opp. to them; and, in like manner, complete the solid ▱ EHPO contained by the three

▱^{s} DEFP, HEFR, DEHX, and those opp. to them. Then,



∵ pyr^d ABCG is sim^r to pyr^d DEFH,

∴ ∠ ABC = ∠ DEF, ∠ GBC = ∠ HEF, Def. 11. 11.

and ∠ ABG = DEH ;

and AB : BC :: DE : EF, Def. 1. 6.

i. e. the sides about the equal ∠^{s} are ::^{ls} ;

∴ ▱ BM is sim^r to EP :

for the same reason,

▱ BN is sim^r to ER, and BK to EX :

∴ the three ▱^{s} BM, BN, BK are sim^r to the three EP, ER, EX :

but the three BM, BN, BK are equal and sim^r to the three w^h are opp. to them, and the three EP, ER, EX, equal and sim^r to the three opp. to them : 24. 11.

∴ the solids BGML, EHPO are contained by the same n^o of sim^r planes ; B. 11.

and their solid angles are equal ;

∴ the solid BGML is sim^r to the solid EHPO : Def. 1 11.

but sim^r solid ▱^{s} have the tripl. r^o of that w^h their homol. sides have ; 33. 11.

∴ the solid BGML has to EHPO the tripl. r^o of that w^h the side BC has to the homol. side EF :

28. 11. but since the prism, w^{h} is the half of the solid ▱
7. 12. is triple of the pyr^{d},

∴ the pyr^{ds} are the sixth part of the solids;

and ∴ pyr^{d} ABCG : pyr^{d} DEFH
15. 5. :: solid BGML : solid EHPO;

∴ *also the pyramid* ABCG *has to the pyramid* DEFH, *the triplicate ratio of that which* BC *has to the homologous side* EF. [Q. E. D.]

Cor.—From this it is evident, that sim^{r} pyr^{ds} w^{h} have multangular bases are likewise to one another in the tripl. r^{o} of their homol. sides.

For the sim^{r} polygons, w^{h} are the bases, may be div^{d} into the same n^{o} of sim^{r} △^{s} homol. to the whole polygons, and the pyr^{ds} so be div^{d} into sim^{r} pyr^{ds} having triangular bases:

∴ as one of the triangular pyr^{ds} in the first multangular pyr^{d} is to one of the triangular pyr^{ds} in the other, so are all the triangular pyr^{ds} in the first to
12. 5. all the triangular pyr^{ds} in the other,
i. e. so is the first multangular pyr^{d} to the other:

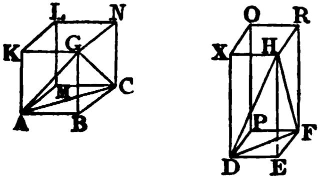
but one triangular pyr^{d} is to its sim^{r} triangular pyr^{d} in the tripl. r^{o} of their homol. sides;
and ∴ the first multangular pyr^{d} has to the other the tripl. r^{o} of that w^{h} one of the sides of the first has to the homol. side of the other.

PROP. IX. THEOR.

The bases and altitudes of equal pyramids having triangular bases are reciprocally proportional; and triangular pyramids, of which the bases and altitudes are reciprocally proportional, are equal to one another.

Let the pyr^{ds} of w^h the $\triangle^s$ ABC, DEF are the bases, and w^h have their vertices in the p^{ts} G, H, be = one another; the bases and altit^s of the pyr^{ds} ABCG, DEFH, shall be reciprocally $::^l$, viz.

base ABC : base DEF
:: altit. of pyr^d DEFH : altit. of pyr^d ABCG.



Complete the ▱^{s} AC, AG, GC, DF, DH, HF; and the solid ▱^{s} BGML, EHPO, contained by these planes and those w^h are opp. to them: then

∵ pyr^d ABCG = pyr^d DEFH,

and that

the solid BGML is sextuple of the pyr^d ABCG, 28. 11.

and the solid EHPO sextuple of the pyr^d DEFH;

Ax. 1. 5. $\therefore$ the solid BGML = the solid EHPO:

but the bases and altit^s of equal solid ⊞^{s} are

34. 11. reciprocally $::^1$;

$\therefore$ base BM : base EP

$::$ altit. of solid EHPO : altit. of solid BGML:

but $\triangle$ ABC : $\triangle$ DEF $::$ base BM : base EP;

15. 5. $\therefore$ $\triangle$ ABC : $\triangle$ DEF

$::$ altit. of solid EHPO : altit. of solid BGML:

but the altit^s of the solid EHPO and the pyr^d DEPH are the same,

as are also the altit^s of the solid BGML and the pyr^d ABCG:

$\therefore$ base ABC : base DEF

$::$ altit. of pyr^d DEFH : altit. of pyr^d ABCG:

$\therefore$ *the bases and altitudes of the pyramids* ABCG, DEFH, *are reciprocally proportional.*

Again, let the bases and altit^s of the pyr^{ds} ABCG, DEFH, be reciprocally $::^1$, viz.

base ABC : base DEF

$::$ altit. of pyr^d DEFH : altit. of pyr^d ABCG:

then shall pyr^d ABCG = pyr^d DEFH.

The same construction being made,

$\because$ base ABC : base DEF

$::$ altit. of pyr^d DEFH : altit. of pyr^d ABCG;

and base ABC : base DEF $::$ ▱ BM : ▱ EP;

$\therefore$ ▱ BM : ▱ EP

$::$ altit. of pyr^d DEFH . altit. of pyr^d ABCG:

but the pyr[d] DEFH and the solid EHPO are of the same altit.,
as are also the pyr[d] ABCG and the solid BGML;
∴ base BM : base EP
:: altit. of ▱ EHPO : altit. of ▱ BGML :
but solid ▱s, which have their bases and altits reciprocally ::[1], are = one another; 34. 11.
∴ solid ▱ BGML = solid ▱ EHPO :

and
the pyr[d] ABCG is the sixth part of the solid BGML,

and
the pyr[d] DEFH is the sixth part of the solid EHPO;
∴ pyr[d] ABCG = pyr[d] DEFH. Ax. 2.

∴ *the bases, &c.* [Q. E. D.]

PROP. X. THEOR.

Every cone is the third part of a cylinder which has the same base and is of an equal altitude with it.

Let a cone and a cyl. have the same base, viz. the ⊙ ABCD, and be of the same altit.: the cone shall be the third part of the cyl., *i. e.* the cyl. shall be triple of the cone.

If the cyl. be not triple of the cone, it must be either > or < the triple.

First, let it be > the triple, and insc. the sq. ABCD in the ⊙:

this sq. is > * the half of the ⊙ ABCD.

On the sq. ABCD erect a prism of the same altit. with the cyl.:

this prism shall be > half of the cyl.:

for let a sq. be desc[d] about the ⊙, and let a prism be erected on the sq. of the same altit. with the cyl.;

then the insc[d] sq. is half of that circumsc[d]:

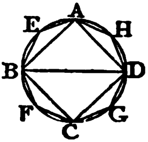
and on these sq. bases are erected solid ▱s, viz. the prisms of the same altit.;

34. 11. and these prisms are to one another as their bases:

∴ the prism on the sq. ABCD is the half of the prism on the sq. desc[d] about the ⊙:

and the cyl. is < the prism on the sq. desc[d] about the ⊙ ABCD:

∴ the prism on the sq. ABCD of the same altit. with the cyl., is > half of the cylinder.



Bis[t] the arcs AB, BC, CD, DA, in the p[ts] E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA: then, as was shown in Prop. 2. XII. each of the △s AEB, BFC, CGD, DHA, is > the half of the seg[t] of the ⊙ in w[h] it stands.

Erect prisms on each of these △s, of the same altit. with the cyl.: each of these prisms shall be > half of the seg[t] of the cyl. in w[h] it is;

* As was shown in Prop. 2. of this book.

[redacted]

for, if through the p^ts^ E, F, G, H, $\|$^s^ be drawn to AB, BC, CD, DA, and ▱^s^ be completed on the same AB, BC, CD, DA, and solid ⊞^s^ be erected on the ▱^s^, the prisms on the △^s^ AEB, BFC, CGD, DHA, are the halves of the solid ⊞^s^: Cor. 7. 12.
and the seg^ts^ of the cyl. w^h^ are on the seg^ts^ of the ⊙ cut off by AB, BC, CD, DA, are $<$ the solid ⊞^s^ w^h^ contain them:

∴ the prisms on the △^s^ AEB, BFC, CGD, DHA, are $>$ half of the seg^ts^ of the cyl. in w^h^ they are:

∴ if each of the arcs be bis^d^, and $|$^s^ be drawn from the p^ts^ of division to the extr^s^ of the arcs, and on the △^s^ thus made, prisms be erected of the same altit. with the cyl., and so on, there must at length remain some seg^ts^ of the cyl., w^h^ together are Lem.
$<$ the excess of the cyl. above the triple of the cone:

let them be those on the seg^ts^ of the ⊙ AE, EB, BF, FC, CG, GD, DH, HA;

∴ the rest of the cyl., *i.e.* the prism of w^h^ the base is the polygon AEBFCGDH, and of w^h^ the altit. is the same with that of the cyl., is $>$ the triple of the cone;

but this prism is triple of the pyr^d^ on the same base, Cor.
of w^h^ the vertex is the same with the vertex of 7. 12.
the cone;

∴ the pyr^d^ on the base AEBFCGDH, having the same vertex with the cone, is $>$ the cone, of w^h^ the base is the ⊙ ABCD:

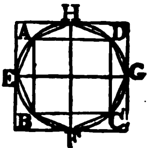
but the pyr^d^ is contained within the cone, and ∴ is $<$ the cone:

w^h^ is impossible:

∴ the cyl. is $\not>$ the triple of the cone.

Nor can the cyl. be $<$ the triple of the cone.
For, if possible, let it be less:
∴ inv^{ly}, the cone is $>$ the third part of the cyl.

In ⊙ ABCD insc. a sq.: this sq. is $>$ half of the ⊙: and on the sq. ABCD erect a pyr^{d}, having the same vertex with the cone; this pyr^{d} is $>$ half of the cone: for, as was before dem^{d},



if a sq. be desc^{d} about the ⊙, the sq. ABCD is the half of it: and if on these sq^{s} there be erected solid ▱^{s} of the same altit. with the cone, w^{h} are also prisms, the prism on the sq. ABCD is the half of that w^{h} is on the sq. desc^{d} about the ⊙; for they are to one another as their bases: as are

2. 11. also the third parts of them:

∴ the pyr^{d} the base of w^{h} is the sq. ABCD, is half of the pyr^{d} on the sq. desc^{d} about the ⊙:

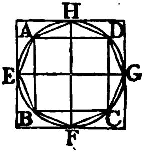
but this last pyr^{d} is $>$ the cone w^{h} it contains;
∴ the pyr^{d} on the sq. ABCD, having the same vertex with the cone, is $>$ the half of the cone.

Bis^{t} the arcs AB, BC, CD, DA in the p^{ts} E, F, G, H, and join AE, EB, BF, FC, CG, GD, DH, HA; then each of the △^{s} AEB, BFC, CGD, DHA, is $>$ half of the seg^{t} of the ⊙ in w^{h} it is; on each of these △^{s} erect pyr^{ds} having the same vertex with the cone:

∴ each of these pyr^{ds} is $>$ the half of the seg^{t} of the cone in w^{h} it is, as before was dem^{d} of the prism and seg^{ts} of the cyl.: and thus bis^{g} each of the arcs, and joining the p^{ts} of division and their extr^{s} by |^{s}, and on the △^{s} erecting pyr^{ds} having their

vertices the same with that of the cone, and so on, there must at length remain some seg[ts] of the cone, w[h] together are $<$ the excess of the cone above the third part of the cyl.: Lemma

let these be the seg[ts] on AE, EB, BF, FC, CG, GD, DH, HA:



$\therefore$ the rest of the cone, *i.e.* the pyr[d] of w[h] the base is the polygon AEBFCGDH, and of w[h] the vertex is the same with that of the cone, is $>$ the third part of the cyl.:

but this pyr[d] is the third part of the prism on the same base AEBFCGDH, and of the same altit. with the cyl.;

$\therefore$ this prism is $>$ the cyl. of w[h] the base is $\odot$ ABCD:

but the prism is contained within the cyl. and $\therefore$ is $<$ the cyl.;
w[h] is impossible:
$\therefore$ the cyl. is $\not<$ the triple of the cone.

And it has been dem[d] that
it is $\not>$ the triple of the cone;
$\therefore$ the cyl. is triple of the cone,
i. e. the cone is the third part of the cyl.

$\therefore$ *every cone, &c.* [Q. E. D.]

PROP. XI. THEOR.

Cones and cylinders of the same altitude, are to one another as their bases.

Let the cones and cyl[s], of w[h] the bases are the ⊙[s] ABCD, EFGH, and the axes KL, MN, and AC, EG the diam[rs] of their bases, be of the same altit.:

cone AL : cone EN :: ⊙ ABCD : ⊙ EFGH.

For, if it be not so, then

⊙ ABCD : ⊙ EFGH

:: cone AL : some solid either > or < cone EN.

Let this solid be X, and first let X be < EN, and let Z be another solid such that Z = EN − X;

∴ cone EN = X + Z.

In ⊙ EFGH insc. the sq. EFGH;

∴ this sq. is > half of the ⊙:

on the sq. EFGH erect a pyr[d] of the same altit. with the cone;

this pyr[d] shall be > half the cone:

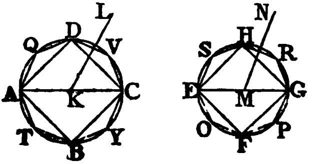
for, if a sq. be desc[d] about the ⊙, and a pyr[d] be erected on it, having the same vertex with the cone*, the pyr[d] insc[d] in the cone is half of the pyr[d] circumsc[d] about it,

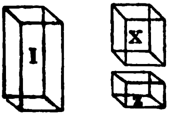
6. 12. (for these pyr[ds] are to one another as their bases);

* Vertex is put in the place of altitude, which is in the Greek, because the pyramid, in what follows, is supposed to be circumscribed about the cone, and so must have the same vertex. And the same change is made in some places following.

but the cone is $<$ the circumscr^{d} pyr^{d};
$\therefore$ the pyr^{d} of w^{h} the base is the sq. EFGH, and its vertex the same with that of the cone, is $>$ half of the cone.

Bis^{t} each of the arcs EF, FG, GH, HE in the p^{ts} O, P, R, S, and join EO, OF, FP, PG, GR, RH, HS, SE:





$\therefore$ each of the $\triangle^{s}$ EOF, FPG, GRH, HSE, is $>$ half of the seg^{t} of the $\odot$ in w^{h} it is:
on each of these $\triangle^{s}$ erect a pyr^{d} having the same vertex with the cone;
each of these pyr^{ds} is $>$ half of the seg^{t} of the cone in w^{h} it is:
and thus bis^{g} each of these arcs, and from the p^{ts} of division drawing $|^{s}$ to the extr^{s} of the arcs, and on each of the $\triangle^{s}$ thus made erecting pyr^{ds} having the same vertex with the cone, and so on, there must at length remain some seg^{ts} of the cone w^{h} are together $<$ the solid Z:
let these be the seg^{ts} on EO, OF, FP, PG, GR, RH, HS, SE:

Lemma.

$\therefore$ the rem^{r} of the cone, viz. the pyr^{d} of w^{h} the base is the polygon EOFPGRHS, and its vertex the same with that of the cone, is $>$ the solid X.

In $\odot$ ABCD insc. the polgon ATBYCVDQ sim^{r} to the polygon EOFPGRHS, and on it erect a pyr^{d} having the same vertex with the cone AL:

1.12. $\because$ polygon ATBYCVDQ : EOFPGRHS

$:: AC^2 : EG^2$;

2.12. and also $\odot$ ABCD : $\odot$ EFGH $:: AC^2 : EG^2$;

$\therefore$ polygon ATBYCVDQ : polygon EOFPGRHS

$::$ $\odot$ ABCD : $\odot$ EFGH :

but $\odot$ ABCD : $\odot$ EFGH $::$ cone AL : solid X;

11.5. and polyg^{n} ATBYCVDQ: polyg^{n} EOFPGRHS, as the pyr^{d} of w^{h} the base is the first of these poly-

6.12. gons, and vertex L, is to the pyr^{d} of w^{h} the base is the other polygon, and its vertex N:

$\therefore$ as the cone AL is to the solid X, so is the pyr^{d} of w^{h} the base is the polygon ATBYCVDQ, and vertex L, to the pyr^{d} the base of w^{h} is the polygon

14.5. EOFPGRHS, and vertex N;

but the cone AL is $>$ the pyr^{d} contained in it:

$\therefore$ the solid X is $>$ the pyr^{d} in the cone EN:

but, as was shown,

X is $<$ that pyr^{d}:

w^{h} is absurd:

$\therefore$ it is not possible that

$\odot$ ABCD : $\odot$ EFGH

$::$ cone AL : any solid $<$ cone EN.

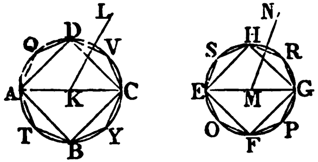
In the same manner it may be dem^{d} to be impossible that

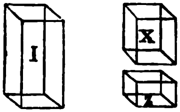
$\odot$ EFGH : $\odot$ ABCD

$::$ cone EN : any solid $<$ cone AL.

Nor is it possible that

⊙ ABCD : ⊙ EFGH
∷ cone AL : any solid > cone EN





For, if possible, let this solid be I, wh is > cone EN. Then invly,

⊙ EFGH : ⊙ ABCD ∷ solid I : cone AL:
but ∵ the solid I is > the cone EN, 14. 5.
∴ solid I : cone AL
∷ cone EN : some solid wh must be < cone EN.
∴ ⊙ EFGH : ⊙ ABCD
∷ cone EN : some solid wh is < cone EN
but this was shown to be impossible:
∴ it is not possible that
⊙ ABCD : ⊙ EFGH
∷ cone AL : any solid > cone EN.

And it has been shown also that it is not possible that

⊙ ABCD : ⊙ EFGH
∷ cone AL : any solid < cone EN:
∴ ⊙ ABCD : ⊙ EFGH ∷ cone AL : cone EN: 15. 5.
but, ∵ the cyls are triple of the cones, 10. 12.

∴ the cyls are to each other as the cones;
and ∴ as ⊙ ABCD is to ⊙ EFGH,
so are the cyls upon them of the same altit.

∴ *cones and cylinders of the same altitude are to one another as their bases.* [Q. E. D.]

PROP. XII. THEOR.

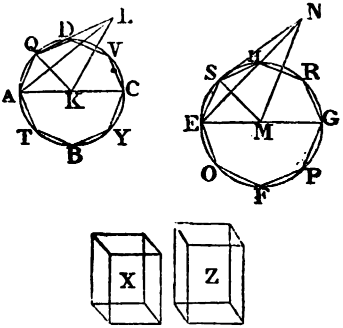
Similar cones and cylinders have to one another the triplicate ratio of that which the diameters of their bases have.

Let the cones and cyls of wh the bases are the ⊙s ABCD, EFGH, and the diamrs of the bases AC, EG, and KL, MN, the axes of the cones or cyls, be simr: the cone of wh the base is the ⊙ ABCD, and vertex the pt L, shall have to the cone of wh the base is the ⊙ EFGH, and vertex N, the tripl. ro of that wh AC has to EG.

For if the cone ABCDL has not to the cone EFGHN the tripl. ro of that which AC has to EG, the cone ABCDL must have the tripl. of that ro to some solid wh is $<$ or $>$ the cone EFGHN.

First, let it have it to a less, viz. the solid X. Make the same constrn as in the preceding propn: it may then be demd in the same way as in that propn, that the pyrd of wh the base is the polygon EOFPGRHS, and vertex N, is $>$ the solid X.

Inscribe also in the ⊙ ABCD the polygon ATBYCVDQ simr to the polygon EOFPGRHS, on wh erect a pyrd having the same vertex with the cone; and let LAQ be one of the $\triangle$s containing the pyrd on the polygon ATBYCVDQ, the vertex of wh is L; and let NES be one of the $\triangle$s containing the pyrd on the polygon EOFPGRHS, of wh the vertex is N; and join KQ, MS: then



$\because$ the cone ABCDL is simr to the cone EFGHN. 24. Def. 11.

$\therefore$ AC : EG :: axis KL : axis MN; 15. 5.

and AC : EG :: AK : EM;

$\therefore$ KL : MN :: AK : EM;

and, altly,

AK : KL :: EM : MN:

and these :: l sides are about the rt $\angle$s AKL, EMN,

$\therefore$ $\triangle$ AKL is simr to $\triangle$ EMN. . 6.

Again,

$\because$ AK : KQ :: EM : MS,

and that these sides are about equal $\angle^s$ AKQ, EMS (for these $\angle^s$ are, each of them, the same part of four $r^t \angle^s$ at the cent^s K, M);

6. 6. $\therefore \triangle$ AKQ is sim^r to $\triangle$ EMS.

And, $\because$, from above,

$$AK : KL :: EM : MN,$$

and that $AK = KQ$, and $EM = MS$;

$$\therefore KQ : KL :: MS : MN:$$

and these sides are about the $r^t \angle^s$ QKL, SMN;

$\therefore \triangle$ LKQ is sim^r to $\triangle$ NMS.

And

$\because \triangle^s$ AKL, EMN are sim^r,

as also $\triangle^s$ AKQ, EMS;

$$\therefore \quad LA : AK :: NE : EM,$$

and $AK : AQ :: EM : ES$;

22. 5. $\therefore$ *ex æq.* $LA : AQ :: NE : ES.$

Again $\because \triangle^s$ LQK, NSM are sim^r

as also $\triangle^s$ KAQ, MES,

$$\therefore LQ : QK :: NS : SM,$$

and $QK : QA :: SM : SE$;

22. 5. $\therefore$ *ex æq.* $LQ : QA :: NS : SE$;

and it was proved that

$$QA : AL :: SE : EN$$

$\therefore$ again, *ex æq.*

$$LQ : AL :: NS : EN:$$

thus, in $\triangle^s$ LQA, NSE, the sides about all the $\angle^s$ are $::^{ls}$,

5. 6. and $\therefore$ the $\triangle^s$ are equiang^r and sim^r to one another: $\therefore$ the pyr^d of w^h the base is the $\triangle$AKQ, and vertex L, is sim^r to the pyr^d the base of w^h is the $\triangle$ EMS, and vertex N, for their solid angles are

= one another, and they are contained by the same n^o of sim^r planes: B. 11.

but sim^r pyr^{ds} w^h have triangular bases have to one another the tripl. r^o of that w^h their homol. sides have; 8. 12.

∴ pyr^d AKQL has to pyr^d EMSN the tripl. r^o of that w^h AK has to EM.

In the same manner, if ⊥s be drawn from the p^{ts} D, V, C, Y, B, T, to K, and from the p^{ts} H, R, G, P, F, O, to M, and pyr^{ds} be erected on the △s having the same vertices with the cones, it may be dem^d that each pyr^d in the first cone has to each in the other, taking them in the same order, the tripl. r^o of that w^h the side AK has to the side EM, *i. e.* w^h AC has to EG:

but as one antecedent is to its consequent, so are all the antecedents to all the consequents;

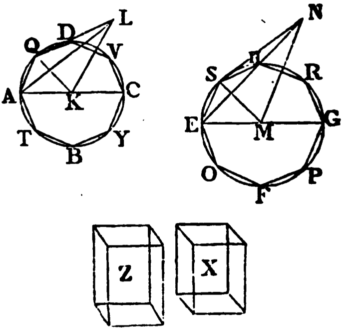
∴ as the pyr^d AKQL to the pyr^d EMSN, so is the whole pyr^d the base of w^h is the $polyg^n$ DQATBYCV, and vertex L, to the whole pyr^d of w^h the base is the polygon HSEOFPGR, and vertex N: 12. 5.

∴ also the first of these two last-named pyr^{ds} has to the other the tripl. r^o of that w^h AC has to EG;

but, by hyp^t, the cone of w^h the base is the ⊙ ABCD, and vertex L, has to the solid X, the tripl. r^o of that w^h AC has to EG;

∴ as the cone of w^h the base is the ⊙ ABCD, and vertex L is to the solid X, so is the pyr^d the base of w^h is the polygon DQATBYCV, and vertex L, to the pyr^d the base of w^h is the polygon HSEOFPGR, and vertex N:

but the said cone is > the pyr^d contained in it;



4. b. ∴ the solid X is > the pyr^{d} the base of w^{h} is the polygon HSEOFPGR, and vertex N:
but it is also less;
w^{h} is impossible:

∴ the cone, of w^{h} the base is the ⊙ ABCD and vertex L, has not to any solid w^{h} is < the cone of w^{h} the base is the ⊙ EFGH and vertex N, the tripl. r^{o} of that w^{h} AC has to EG.

In the same manner it may be dem^{d}, that neither has the cone EFGHN to any solid w^{h} is < the cone ABCDL, the tripl. r^{o} of that w^{h} EG has to AC.

Nor can the cone ABCDL have to any solid w^{h} is > the cone EFGHN, the tripl. r^{o} of that w^{h} AC has to EG.

For, if possible, let it have it to a greater, viz. the solid Z:

∴ inv^{ly}, the solid Z has to the cone ABCDL the

[redacted]

tripl. r^{o} of that which EG has to AC:
but $\because$ the solid Z is $>$ the cone EFGHN,
$\therefore$ as the solid Z is to the cone ABCDL,
so is the cone EFGHN to some solid,
w^{h} must be $<$ the cone ABCDL ; 14. 5.
$\therefore$ the cone EFGHN has to a solid w^{h} is $<$ the cone ABCDL the tripl. r^{o} of that w^{h} EG has to AC,
w^{h} was dem^{d} to be impossible:
$\therefore$ the cone ABCDL has not to any solid $>$ the cone EFGHN, the tripl. r^{o} of that w^{h} AC has to EG: and it was dem^{d} that it could not have that r^{o}
to any solid $<$ the cone EFGHN,
$\therefore$ the cone ABCDL has to the cone EFGHN, 15. 5.
the tripl. r^{o} of that w^{h} AC has to EG:
but every cone is the third part of the cyl. on the 10. 12
same base, and of the same altit.:
and $\therefore$ as the cones are to each other,
so are the corresponding cyl^{s}:
$\therefore$ also the cyl. has to the cyl.
the tripl. r^{o} of that w^{h} AC has to EG.

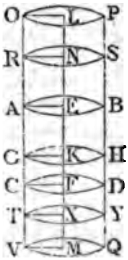
$\therefore$ *similar cones, &c.* [Q. E. D.]

PROP XIII. THEOR.

If a cylinder be cut by a plane parallel to its opposite planes, or bases, it divides the cylinder into two cylinders, one of which is to the other as the axis of the first to the axis of the other.

Let the cyl. AD be cut by the plane GH $\parallel$

the opp. planes AB, CD, meeting the axis EF in the p^t K, and let the line GH be the com. section of the plane GH and the surface of the cyl. AD.



Let AEFC be the ▱ in any position of it, by the revolution of w^h about the | EF the cyl. AD is $desc^d$; and let GK be the com. section of the plane GH, and the plane AEFC.

Then, ∵
the || planes AB, GH are cut by the plane AEKG,
∴ AE, KG, their com. sections with it, are || :
16. 11. ∴ AK is a ▱,
and GK = EA, the | from the cent. of the ⊙ AB:
In the same manner each of the $|^s$ drawn from the p^t K to the line GH may be proved to be = those w^h are drawn from the cent. of ⊙ AB to its $⊙^{ce}$
15Def.1 and ∴ these $|^s$ are all = one another;
∴ the line GH is the $⊙^{ce}$ of a ⊙ of w^h the cent. is the p^t K:
∴ the plane GH div^s the cyl. AD into the cyl^s AH, GD;
for they are the same w^h would be $desc^d$ by the revolution of the $▱^s$ AK, GF about the $|^s$ EK, KF, and it is to be shown, that

cyl. AH : cyl. HC :: axis EK : axis KF

Prod. the axis EF both ways: and take any n^o of $|^s$ EN, NL, each = EK; and any n^o FX, XM, each = FK; and let planes || AB, CD, pass through the p^{ts} L, N, X, M;

$\therefore$ as was proved of the plane GH, the com. sections of these planes with the cyl. prod. are ⊙s, the cents of wh are the pts L, N, X, M ; and these planes cut off the cyls PR, RB, DT, TQ. And,

$\because$ axis $LN = NE = EK$

$\therefore$ the cyls PR, RB, BG, are to one another as their bases : but their bases are equal, and $\therefore$ the cyls PR, RB, BG are equal :

And,
$\because$ the axes LN, NE, EK are = one another, as also the cyls PR, RB, BG,

and that there are as many axes as cyls ;

$\therefore$ whatever mult. the axis KL is of the axis KE, the same mult. is the cyl. PG of the cyl. GB :

for the same reason,
whatever mult. the axis MK is of the axis KF, the same mult. is the cyl. QG of the cyl. GD:

but as the axis KL is $>$, $=$ or $<$ the axis KM :

so is the cyl. PG $>$, $=$ or $<$ the cyl. GQ :

$\therefore$ since there are four magns, viz. the axes, EK, KF, and the cyls BG, GD : and that of the axis EK and cyl. BG

there have been taken any equimults whatever,

viz. the axis KL and cyl. PG ;

and of the axis KF and cyl. GD any equimult whatever, viz. the axis KM and cyl. GQ ;
and since also it has been demd that
as the axis KL is $>$, $=$ or $<$ the axis KM,

so the cyl. PG is >, = or < the cyl. GQ;
∴ cyl. BG : cyl. GD :: axis EK : axis KF.

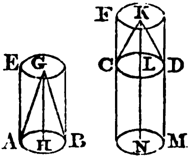
∴ *if a cylinder, &c.* [Q. E. D.]

PROP. XIV. THEOR.

Cones and cylinders upon equal bases are to one another as their altitudes.

Let the cyl^s EB, FD, be on equal bases AB, CD
cyl. EB : cyl. FD :: axis GH : axis KL.

Prod. the axis KL to the p^t N, making LN = the axis GH, and let CM be a cyl. of w^h the base is CD, and axis LN. Then,



∵ the cyl^s EB, CM have the same altit.
11. 12. ∴ they are to one another as their bases;
but their bases are equal,
∴ also the cyl^s EB, CM, are equal.

And,
∵ the cyl. FM is cut by the plane CD
∥ to its opp. planes;

∴ cyl. CM : cyl. FD :: axis LN : axis KL : 12. 12.
but the cyl. CM = the cyl. EB,
and the axis LN = the axis GH ; 15. 5.
∴ cyl. EB : cyl. FD :: axis GH : axis KL :

And
∵ the cyl[s] are triple of the cones, 10. 12.
∴ cone ABG : cone CDK :: cyl. EB : cyl. FD :
∴ also
cone ABG : cone CDK :: axis GH : axis KL.

∴ *cones and cylinders, &c.* [Q. E. D.]

PROP. XV. THEOR.

The bases and altitudes of equal cones and cylinders are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the ⊙[s] ABCD, EFGH, the diam[rs] of w[h] are AC, EG, be the bases, and KL, MN, the axes, as also the altit[s], of equal cones and cyl[s]; and let ALC, ENG be the cones, and AX, EO the cyl[s]: the bases and altit[s] of the cyl[s] AX, EO, shall be reciprocally ::[l], viz.

base ABCD : base EFGH :: altit. MN : altit. KL.

The altit[s] MN, KL must either be equal, or be unequal.

First, let them be equal: then

since the cyl^s AX, EO are also equal,

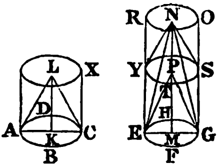
and that cones and cyl^s of the same altit. are to

11. 12. one another as their bases;

A. 5. $\therefore$ the base ABCD = the base EFGH;

and $\therefore$

base ABCD : base EFGH :: altit. MN : altit. KL.



But let the altit^s KL, MN, be unequal; and MN being the greater of the two, take from it MP = KL, and through the p^t P cut the cyl. EO by the plane TYS, || the opp. planes of the ⊙^{s} EFGH, RO: then, the com. section of the plane TYS and the cyl. EO is a ⊙, and

$\therefore$ ES is a cyl., the base of w^h is the ⊙ EFGH, and altit. MP:

And,

$\because$ the cyl. AX = the cyl. EO,

7. 5. $\therefore$ cyl. EO : cyl. ES :: AX : the same ES:

but since the cyl^s AX, ES are of the same altit.

11. 12. $\therefore$ cyl. AX : cyl. ES :: base ABCD : base EFGH;

and $\because$ the cyl. EO is cut by the plane TYS

|| its opp. planes,

$\therefore$ cyl. EO : cyl. ES :: altit. MN : MP or KL ; 13. 12
$\therefore$ base ABCD : base EFGH
:: altit. MN : altit. KL ;

i. e. the bases and altit[s] of the equal cyl[s] AX, EO
are reciprocally ::[1].

But let the bases and altit[s] of the cyl[s] AX, EO, be reciprocally ::[1], viz.

base ABCD : base EFGH :: altit. MN : altit. KL :
then shall the cyl. AX = the cyl. EO.

First, let the base ABCD = the base EFGH :
then, $\because$
base ABCD : base EFGH :: altit. MN : altit. KL;
$\therefore$ MN = KL : A. 5.
and $\therefore$ cyl. AX = cyl. EO. 11. 12.

But let the bases ABCD, EFGH be unequal, and let ABCD be the greater of the two ; whence,
$\because$ base ABCD : base EFGH
:: altit. MN : altit. KL,
$\therefore$ MN is > KL. A. 5.

Then, the same constr[n] being made as before,
$\because$ base ABCD : base EFGH
:: altit. MN : altit. KL ;

and that,
$\because$ altit. KL = altit. MP,
$\therefore$ cyl. AX : cyl. ES 11. 12
:: base ABCD : base EFGH

and also
cyl. EO : cyl. ES
:: altit. MN : altit. MP or KL ;

∴ cyl. AX : cyl. ES
:: cyl. EO : cyl. ES;
whence cyl. AX = cyl. EO:
And the same reasoning holds in cones.

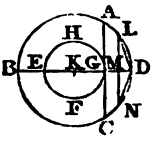
[Q. E. D.]

PROP. XVI. PROB.

In the greater of two circles, that have the same centre, to inscribe a polygon of an even number of equal sides, that shall not meet the less circle.

Let ABCD, EFGH be two given ⊙ˢ having the same cent. K: it is req. to insc. in the greater ⊙ ABCD, a polygon of an even nº of equal sides that shall not meet the less ⊙.

Through the cent. K draw the | BD, and from the pᵗ G, where it meets the ⊙ᶜᵉ of the less ⊙, draw GA at rᵗ ∠ˢ to BD, and prod. it to C;
∴ AC touches the ⊙ EFGH:



16. 3. then, if the arc BAD be bisᵈ, and the half of it be again bisᵈ, and so

Lemma on, there must at length remain an arc < AD: let this be LD: from L draw LM ⊥ to BD, prod. it to N; and join LD, DN:
∴ LD = DN.

And, ∵ LN is ∥ AC,

and that AC touches the ⊙ EFGH ;
∴ LN does not meet the ⊙ EFGH ;
and much less shall the |s LD, DN meet the ⊙ EFGH:

so that, *if straight lines, each equal to* LD, *be applied in the circle* ABCD *from the point* L *around to* N, *there shall be inscribed in the circle a polygon of an even number of equal sides not meeting the less circle.* [Q. E. F.]

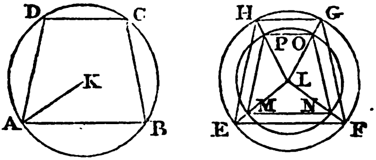
LEMMA II.

If two trapeziums ABCD, EFGH *be inscribed in the circles, the centres of which are the points* K, L ; *and if the sides* AB, DC *be parallel, as also* EF, HG ; *and the other four sides* AD, BC, EH, FG, *be all equal to one another ; but the side* AB *greater than* EF, *and* DC *greater than* HG : *the straight line* KA *from the centre of the circle in which the greater sides are, is greater than the straight line* LE *drawn from the centre to the circumference of the other circle.*

If it be possible, let KA be ⊁ LE ;
then KA must be either = or < LE.

First, let KA = LE :
then the two ⊙s are equal :
and ∵ |s AD, BC = |s EH, FG, each to each,
∴ the arcs AD, BC = the arcs EH, FG, each to each : 28. 3.

but, $\because$ |s AB, DC are $>$ EF, GH, each than each,
$\therefore$ the arcs AB, DC are $>$ EF, GH, each than each;
$\therefore$ the whole $\odot$ ce ABCD is $>$ the whole $\odot$ ce EFGH:
but these $\odot$ ces are also equal,
wh is impossible.
$\therefore$ | KA is $\neq$ LE.



But let KA be $<$ LE; and make LM $=$ KA, and from the cent. L, at dist. LM, desc. $\odot$ MNOP, meeting the |s LE, LF, LG, LH, in M, N, O, P; and join MN, NO, OP PM, wh are respectively
2\. 6. || to and $<$ EF, FG, GH, HE: then,
$\because$ EH is $>$ MP,
$\therefore$ AD is $>$ MP;
and the $\odot$s ABCD, MNOP are equal;
$\therefore$ arc AD is $>$ MP:

for the same reason,
arc BC is $>$ NO:
and $\because$ | AB is $>$ EF, wh is $>$ MN,
à fortiori, $\therefore$ AB is $>$ MN:
$\therefore$ arc AB is $>$ MN;

and for the same reason,
arc DC is $>$ PO:
$\therefore$ the whole $\odot$ ce ABCD is $>$ the whole MNOP:
but these $\odot$ ces are likewise equal;
wh is impossible;

∴ KA is $\nless$ LE:
neither is KA = LE;
∴ | KA must be > LE.

[Q. E. D.]

Cor.—And if there be an isosc. △, the sides of wh are = AD, BC, but its base < AB, the greater of the two sides AB, DC; it may, in the same manner, be demd that | KA is > the | drawn from the cent. to the ⊙ce of the ⊙ descd about the △.

PROP. XVII. PROB.

In the greater of two spheres which have the same centre, to inscribe a solid polyhedron, the superficies of which shall not meet the less sphere.

Let there be two spheres about the same cent. A: it is reqd to insc. in the greater a solid polyhedron, the superficies of wh shall not meet the less sphere.

Let the spheres be cut by a plane passing through the cent.; the com. sections of it with the spheres shall be ⊙s; for the sphere is descd by the revolution of a $\frac{1}{2}$ ⊙ about the diamr remg unmoved; so that in whatever position the $\frac{1}{2}$ ⊙ be conceived, the com. section of the plane in wh it is with the superficies of the sphere is the ⊙ce of a ⊙; and this is a great ⊙ of the sphere, for the diamr of the sphere, wh is likewise the diamr of the ⊙, is > any line *in the ⊙ or sphere.*

Let then the ⊙ made by the section of the plane with the greater sphere be BCDE, and with the less sphere be FGH; and draw the two diam^{rs} BD, CE,
16. 12. at r^{t} ∠^{s} to one another; and in BCDE, the greater of the two ⊙^{s}, insc. a polygon of an even n^{o} of equal sides, not meeting the less ⊙ FGH; and let its sides in BE the fourth part of the ⊙, be BK, KL, LM, ME; join KA, and prod. it to N; and from A
12. 1. draw AX at r^{t} ∠^{s} to the plane of the ⊙ BCDE, meeting the superficies of the sphere in the p^{t} X: and let planes pass through AX, and each of the |^{s} BD, KN, w^{h}, from what has been said, shall prod. great ⊙^{s} on the superficies of the sphere, and let BXD, KXN be the $\frac{1}{2}$ ⊙^{s} thus made upon the diam^{rs} BD, KN: then,

∵ XA is at r^{t} ∠^{s} to the plane of the ⊙ BCDE,
18. 11. ∴ every plane w^{h} passes through XA is at r^{t} ∠^{s} to the plane of the ⊙ BCDE;

∴ the $\frac{1}{2}$ ⊙^{s} BXD, KXN are at r^{t} ∠^{s} to that plane:
and ∵ the $\frac{1}{2}$ ⊙^{s} BED, BXD, KXN on the equal diam^{rs} BD, KN, are = one another;

∴ their halves BE, BX, KX are = one another;
and ∴ as many sides of the polygon as are in BE, so many are there in BX, KX, = the sides BK, KL, LM, ME:

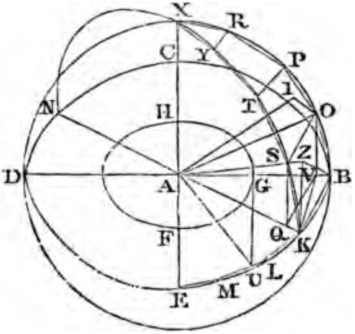
let these polygons be desc^{d}, and their sides be BO, OP, PR, RX; KS, ST, TY, YX;
and join OS, PT, RY; and from the p^{ts} O, S, draw OV, SQ, ⊥^{s} to AB, AK: then,

∵ the plane BOXD is at r^{t} ∠^{s} to the plane BCDE,
and in one of them BOXD, OV is drawn ⊥ to AB, the com. section of the planes,
Def. 4. ∴ OV is ⊥ to the plane BCDE:

for the same reason,

SQ is $\perp$ to the same plane,

for the plane KSXN is at r[t] $\angle$[s] to the plane BCDE.



Join VQ: then,

$\because$ in the equal $\frac{1}{2}$ $\odot$[s] BXD, KXN, the arcs BO, KS are equal, and OV, SQ are $\perp$ to their diam[rs], 26. 1.

$\therefore$ OV = SQ, and BV = KQ:

but the whole BA = the whole KA;

$\therefore$ the rem[r] VA = the rem[r] QA.

$\therefore$ KQ : QA :: BV : VA; 2. 6.

$\therefore$ VQ is $\parallel$ BK:

and,

$\because$ OV, SQ are both at r[t] $\angle$[s] to the plane of the 6. 11.

$\odot$ BCDE,

$\therefore$ OV is $\parallel$ SQ;

and it has also been proved that

OV = SQ;

$\therefore$ QV, SO are equal and $\parallel$: 33. 1.

and, ∵ QV is ∥ SO, and also ∥ KB,

9.11. ∴ SO is ∥ KB;

and ∴ BO, KS, w^h join them, are in the same plane in w^h these ∥^s are, and the quadrilat^l fig. KBOS is in one plane:

and if PB, TK be joined, and ⊥^s be drawn from the p^ts P, T to the |^s AB, AK, it may be dem^d that TP is ∥ KB in the same way that SO was shown to be ∥ the same KB;

9.11. ∴ TP is ∥ SO,

and the quadrilat^l fig. SOPT is in one plane: for the same reason,

the quadrilat^l TPRY is in one plane:

2.11. and the fig. YRX is also in one plane:

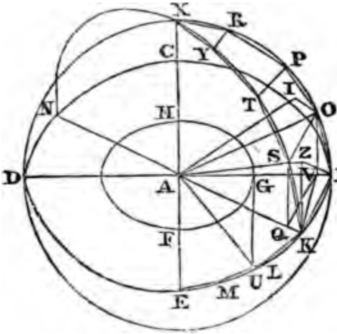
∴ if from the p^ts O, S, P, T, R, Y, there be drawn |^s to the p^t A, there will be formed a solid polyhedron between the arcs BX, KX, composed of pyr^ds, the bases of w^h are the quadrilat^s KBOS, SOPT, TPRY, and the △ YRX, and of w^h the com. vertex is the p^t A:

and if the same constr^n be made on each of the sides KL, LM, ME, as has been done upon BK, and the like be done also in the other three quadrants, and in the other hemisphere; there will be formed a solid polyhedron insc^d in the sphere, composed of pyr^ds, the bases of w^h are the aforesaid quadrilat^l fig^s, and the △ YRX, and those formed in the like manner, in the rest of the sphere, the com. vertex of them all being the p^t A.

Also the superficies of this solid polyhedron shall

11.11. not meet the less sphere in w^h is the ⊙ FGH.

For, from the p[t] A draw AZ $\perp$ to the plane of the quadrilat[l] KBOS, meeting it in Z, and join BZ, ZK: then,



$\because$ AZ is $\perp$ to the plane KBOS,

$\therefore$ it makes r[t] $\angle$[s] with every | meeting it in that plane;

$\therefore$ AZ is $\perp$ to BZ and ZK:

and, $\because$ $AB = AK$,

and that $AB^2 = AZ^2 + ZB^2$,

and $AK^2 = AZ^2 + ZK^2$; 47. 1.

$\therefore AZ^2 + ZB^2 = AZ^2 + ZK^2$;

and AZ^2 being taken from these equals,

the rem[r] $ZB^2 =$ the rem[r] ZK^2;

and $\therefore$ | $ZB = ZK$:

In the like manner it may be dem[d] that the |[s] drawn from the p[t] Z to the p[ts] O, S, are = BZ or ZK; $\therefore$ the $\odot$ desc[d] from the cent. Z, and dist. ZB, will pass through the p[ts] K, O, S, and KBOS will be a quadrilat[l] fig. in the $\odot$:

and ∵ KB is $> QV$, and $QV = SO$,
∴ KB is $> SO$:
but KB is $=$ each of the |s BO, KS;
∴ each of the arcs cut off by KB, BO, KS,
is $>$ that cut off by OS;
and these three arcs, together with a fourth $=$ one of them, are $>$ the same three together with that cut off by OS, *i.e.* $>$ the whole ⊙ce of the ⊙;
∴ the arc subtended by KB is $>$ the fourth part of the whole ⊙ce of the ⊙ $KBOS$,
and ∴ ∠ BZK at the cent. is $>$ a rt ∠;
and ∵ ∠ BZK is obt.,
2. 2. ∴ BK^2 is $> BZ^2 + ZK^2$,
i. e. $> 2\,BZ^2$.
Join KV: then, in the △s KBV, OBV,
∵ KB, $BV = OB$, BV, each to each,
and that they contain equal ∠s;
4. 1. ∴ ∠ $KVB =$ ∠ OVB:
and OVB is a rt ∠:
∴ also KVB is a rt ∠:
and ∵ BD is $< 2\,DV$,
a. 6. ∴ the rect. $BD.BV$ is $<$ twice the rect. $BV.DV$;
i.e. KB^2 is $< 2\,KV^2$:
but KB^2 is $> 2\,BZ^2$;
∴ KV^2 is $> BZ^2$:
and ∵ $BA = AK$,
and that $BA^2 = BZ^2 + ZA^2$,
$AK^2 = KV^2 + VA^2$;
∴ $BZ^2 + ZA^2 = KV^2 + VA^2$;
and of these sqs, KV^2 is $> BZ^2$,
∴ ZA^2 is $> VA^2$.
and ZA is $> VA$:
à fortiori, ∴ AZ is $> AG$,

for, in the preceding propn, it was shown that
KV falls without ⊙ FGH ;
and AZ is ⊥ to the plane KBOS,

and ∴ is the shortest of all the |s that can be drawn from A, the cent. of the sphere, to that plane.

∴ the plane KBOS does not meet the less sphere.

And that the other planes between the quadrants BX, KX, fall without the less sphere, is thus demd.

From the pt A draw AI ⊥ to the plane of the quadrilatl SOPT, and join IO: then, as was demd of the plane KBOS and the pt Z, it may similarly be shown that the point I is the cent. of a ⊙ descd about SOPT ; and that OS is > PT ;

and it was shown that PT is || OS ;

hence,
∵ in the two trapeziums KBOS, SOPT inscd in ⊙s the sides BK, OS are ||s, as also OS, PT ; and the other sides BO, KS, OP, ST all = one another,
and that BK is > OS, and OS > PT,
∴ | ZB is > IO.

Join AO ; it will be = AB ;
and ∵ AIO, AZB are rt ∠s, 2 Lem 12.

$$\therefore AI^2 + IO^2 = AO^2$$
$$= AB^2$$
$$= AZ^2 + ZB^2;$$

and ZB^2 is > IO^2;
∴ AZ^2 is < AI^2,
and AZ is < AI :

And it was proved that

$$AZ \text{ is} > AG;$$

à fortiori, $\therefore AI$ is $> AG$:

$\therefore$ the plane SOPT falls wholly without the less sphere.

In the same manner it may be dem[d] that the plane TPRY falls without the same sphere, as also the $\triangle$ YRX, viz. by the Cor. of 2d Lemma.

And similarly it may be dem[d] that all the planes, w[h] contain the solid polyhedron, fall without the less sphere.

$\therefore$ ***in the greater of two spheres, which have the same centre, a solid polyhedron is described, the superficies of which does not meet the less sphere.***

[Q. E. F.]

But that | AZ is $>$ AG, may be dem[d] otherwise, and in a shorter manner, without the help of Prop. 16., as follows.

From the p[t] G draw GU at r[t] $\angle$[s] to AG, and join AU.

If then the arc. BE be bis[d], and its half again bis[d], and so on, there will at length remain an arc $<$ the arc w[h] is subtended by a $| = GU$, insc[d] in the $\odot$ BCDE: let this be the arc KB:

$$\therefore | KB \text{ is} < GU:$$

and $\because$, as was proved in the preceding,

$$\angle BZK \text{ is obt.,}$$

$$\therefore KB \text{ is} > BZ:$$

$$\text{but } GU \text{ is} > KB;$$

à fortiori, $\therefore$ GU is $>$ BZ,

$$\text{and } GU^2 > BZ^2:$$

$$\text{and } AU = AB;$$
$$\therefore AU^2 = AB^2,$$
$$i.\,e.\ AG^2 + GU^2 = AZ^2 + ZB^2;$$
$$\text{but } GU^2 \text{ is} > BZ^2:$$
$$\therefore AZ^2 \text{ is} > AG^2,$$
$$\text{and } \therefore AZ \text{ is} > AG.$$

Cor. And if in the less sphere there be desc[d] a solid polyhedron, by drawing |[s] betwixt the p[ts] in w[h] the |[s] from the cent. of the sphere drawn to all the angles of the solid polyhedron in the greater sphere meet the superficies of the less; in the same order in w[h] are joined the p[ts] in w[h] the same |[s] from the cent. meet the superficies of the greater sphere; the solid polyhedron in the sphere BCDE has to this other solid polyhedron the tripl. r[o] of that w[h] the diam[r] of the sphere BCDE has to the diam[r] of the other sphere:

For if these two solids be div[d] into the same n[o] of pyr[ds], and in the same order, the pyr[ds] shall be sim[r] to one another, each to each; since they have the solid angles at their com. vertex, the cent. of the sphere the same in each pyr[d], and their other solid angles at the bases = one another, each to each, for B. 11.
they are contained by three plane ∠[s] each = each;
and
the pyr[ds] are contained by the same n[o] of sim[r] planes,
and ∴ are sim[r] to one another, each to each: 11. Def. 11.
But sim[r] pyr[ds] have to one another the tripl. r[o] Cor. 12.
of their homol. sides:
∴ the pyr[d] of w[h] the base is the quadrilat[l] KBOS, and vertex A, has to the pyr[d] in the other sphere of the same order, the tripl. r[o] of their homol. sides,

i. e. of that r^o w^h AB from the cent. of the greater sphere has to the | from the same cent. to the superficies of the less sphere.

And, in like manner, each pyr^d in the greater sphere has to each of the same order in the less, the tripl. r^o of that w^h AB has to the semi-$diam^r$ of the less.

And as one antecedent to its consequent, so are all the antecedents to all the consequents.

∴ the whole solid polyhedron in the greater sphere has to the whole solid polyhedron in the other, the tripl. r^o of that w^h AB the semi-$diam^r$ of the first has to the semi-$diam^r$ of the other; *i. e.* w^h the $diam^r$ BD of the greater has to the $diam^r$ of the other sphere.

PROP. XVIII. THEOR.

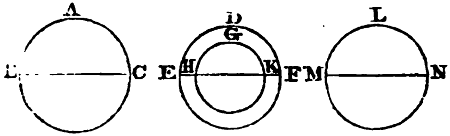
Spheres have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF, be two spheres, of w^h the $diam^{rs}$ are BC, EF The sphere ABC has to the sphere DEF the tripl. r^o of that w^h BC has to EF.

For, if it has not, the sphere ABC shall have to a sphere either $<$ or $>$ DEF, the tripl. r^o of that w^h BC has to EF.

First, let it have that r^o to a less, viz. the sphere GHK; and let the sphere DEF have the same cent. with GHK; and in the greater sphere DEF

desc. a solid polyhedron, the superficies of w^{h} does not meet the less sphere GHK: and in the sphere ABC desc. another, sim^{r} to that in the sphere DEF: 17. 12.



∴ the solid polyhedron in the sphere ABC has to the solid polyhedron in DEF, the tripl. r^{o} of that w^{h} BC has to EF. Cor. 17 12.

But the sphere ABC has to the sphere GHK, the tripl. r^{o} of that w^{h} BC has to EF;

∴ $\left.\begin{array}{l}\text{the polyhedron in}\\ \text{the sphere ABC}\end{array}\right\} : \left\{\begin{array}{l}\text{the polyhedron in}\\ \text{the sphere DEF}\end{array}\right\}$
:: sphere ABC : sphere GHK.

But the sphere ABC is > the solid polyhedron in it:

∴ also the sphere GHK is > the polyhedron in the sphere DEF: 14. 5.

but the sphere is contained within,
and ∴ is also < the polyhedron,
w^{h} is impossible:

∴ the sphere ABC has not to any sphere < DEF, the tripl. r^{o} of that w^{h} BC has to EF.

In the same manner, it may be dem^{d}, that the sphere DEF has not to any sphere < ABC, the tripl. r^{o} of that w^{h} EF has to BC.

Nor can the sphere ABC have to any sphere > DEF, the tripl. r^{o} of that w^{h} BC has to EF.

For, if it can, let it have that r° to a greater sphere LMN :

∴ *inv^ly*, the sphere LMN has to the sphere ABC,
the tripl. r° of that w^h EF has to BC.

But,

∵ the sphere LMN is > the sphere DEF,
∴ as the sphere LMN to ABC,
14. 5. so is the sphere DEF to some sphere < ABC ;
and ∴ the sphere DEF has to a sphere < ABC,
the tripl. r° of that w^h EF has to BC,
w^h was shown to be impossible :
∴ the sphere ABC has not to any sphere > DEF,
the tripl. r° of that w^h BC has to EF ;
and it was dem^d that neither has it that r° to any
sphere < DEF.

∴ *the sphere* ABC *has to the sphere* DEF *the triplicate ratio of that which* BC *has to* EF.

[Q. E. D.]

FINIS.

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